Advantage of an **Extended Network of Detectors**

In Radiometric Searches for GW

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GWB Detection Strategy

- Detector output = true signal + noise

$$\begin{aligned} s_1(t) &= h_1(t) + n_1(t) \\ s_2(t) &= h_2(t) + n_2(t) \end{aligned}$$

- Normally detector noise are uncorrelated: $\langle n_1(t) n_2(t') \rangle = 0$
- SGWB signal is characterized by correlation
- Cross-correlation (CC) statistic is the best choice

$$\langle s_1(t)\, s_2(t') \rangle$$

  - one detector's signal is the filter for other detector's data

Cross Spectral Density (CSD)

- Observed data := Cross Spectral Density := product of SFT’s

$$\mathbf{C}^I \;\equiv\; C^I_{ft} \;:=\; \widetilde{s}^*_{I_1}(t;f)\,\widetilde{s}_{I_2}(t;f)$$

- Noise (in the small signal limit):

$$\mathbf{n}^I \;\equiv\; n^I_{ft} \;:=\; \widetilde{n}^*_{I_1}(t;f)\,\widetilde{n}_{I_2}(t;f)$$

- Covariance matrix:

$$\mathbf{N} \;\equiv\; \mathrm{Cov}(C^I_{ft}, C^{I'}_{f't'}) \;\approx\; \frac{(\Delta T)^2}{4}\,\delta_{II'}\,\delta_{tt'}\,\delta_{ff'}\,P_{I_1}(t;f)\,P_{I_2}(t;f)$$

CSD Generated by an Anisotropic Background

- Anisotropic SGWB in some basis: $\mathcal{P}(\hat{\mathbf{\Omega}}) := \sum_\alpha \mathcal{P}_\alpha \, e_\alpha(\hat{\mathbf{\Omega}}); \quad \boldsymbol{\mathcal{P}} \equiv \mathcal{P}_\alpha$
- Observed CSD = convolution of anisotropic background with additive noise

$$\boxed{C^I_{ft} := \sum_\alpha K^I_{ft,\alpha} \mathcal{P}_\alpha + n^I_{ft}}$$

Low signal limit

  - the “kernel” or “beam”:

$$\mathbf{K}^I \equiv K^I_{ft,\alpha} := \Delta T \, H(f) \, \gamma^I_\alpha(f,t)$$

    - ✳ generalized overlap reduction function

$$\gamma^I_\alpha(f,t) := \sum_{A=+,\times} \int_{S^2} \mathrm{d}\hat{\mathbf{\Omega}} \, F^A_{I_1}(\hat{\mathbf{\Omega}},t) \, F^A_{I_2}(\hat{\mathbf{\Omega}},t) \, e^{2\pi \mathrm{i} f \hat{\mathbf{\Omega}} \cdot \mathbf{\Delta x}(t)/c} \, e_\alpha(\hat{\mathbf{\Omega}})$$

ML Estimation of SGWB Anisotropy

- ML estimates in any basis with a network of detectors:

$$\boxed{\hat{\mathcal{P}}_\alpha \equiv \hat{\boldsymbol{\mathcal{P}}} = \boldsymbol{\Sigma}\cdot\mathbf{X}}$$

  - “Dirty” map (essentially filtered output):

$$\mathbf{X} := \mathbf{K}^\dagger\cdot\mathbf{N}^{-1}\cdot\mathbf{C} \Rightarrow \boxed{X_\alpha = \frac{4}{\Delta T}\sum_{I,ft}\frac{H(f)\,\gamma^{I*}_{ft,\alpha}}{P_{I_1}(t;f)\,P_{I_2}(t;f)}\,\widetilde{s}^*_{I_1}(t;f)\,\widetilde{s}_{I_2}(t;f)}$$

  - Fisher information matrix:

$$\boldsymbol{\Sigma}^{-1} := \mathbf{K}^\dagger\cdot\mathbf{N}^{-1}\cdot\mathbf{K} \Rightarrow \boxed{[\Sigma^{-1}]_{\alpha\alpha'} = 4\sum_{I,ft}\frac{H^2(f)}{P_{I_1}(t;f)\,P_{I_2}(t;f)}\,\gamma^{I*}_{\alpha}(f,t)\,\gamma^{I}_{\alpha'}(f,t)}$$

Specific Cases

- Optimal search
  - model of the sky as one component basis:

$$e_\alpha(\hat{\mathbf{\Omega}}) \;:=\; \mathcal{P}_A(\hat{\mathbf{\Omega}})$$

  - most general **overlap reduction function**:

$$\gamma_{\mathcal{P}_\pm}(t,f) \;:=\; \int_{S^2} \mathrm{d}\hat{\mathbf{\Omega}}_0 \, e^{2\pi \mathrm{i} f \hat{\mathbf{\Omega}}_0 \cdot \mathbf{\Delta x}(t)/c} \sum_{A=\pm} F_1^A(\hat{\mathbf{\Omega}}_0, t)\, F_2^A(\hat{\mathbf{\Omega}}_0, t)\, \mathcal{P}_A(\hat{\mathbf{\Omega}}_0)$$

  - all sky search for anisotropic background
    - requires a good model of the angular power distribution

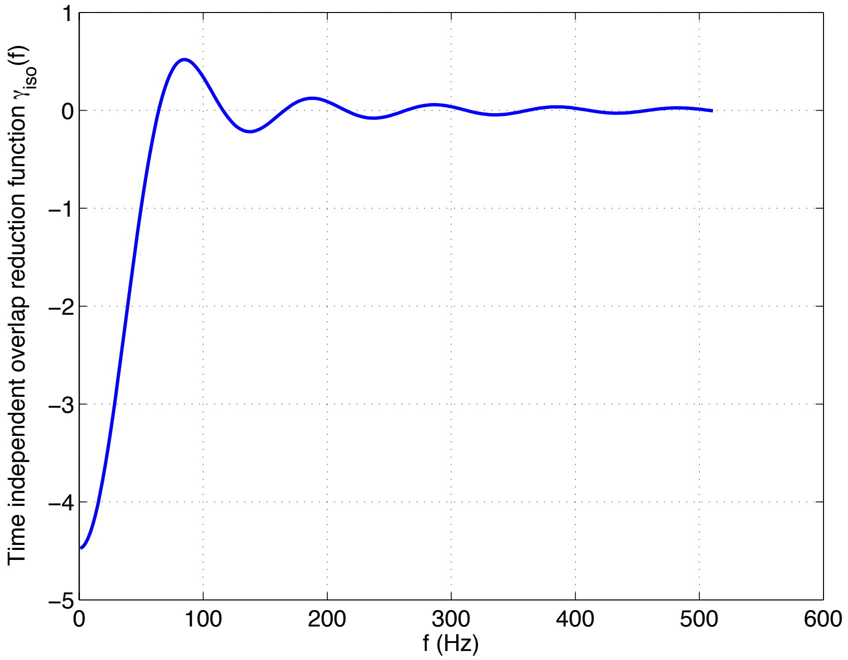
Isotropic Search

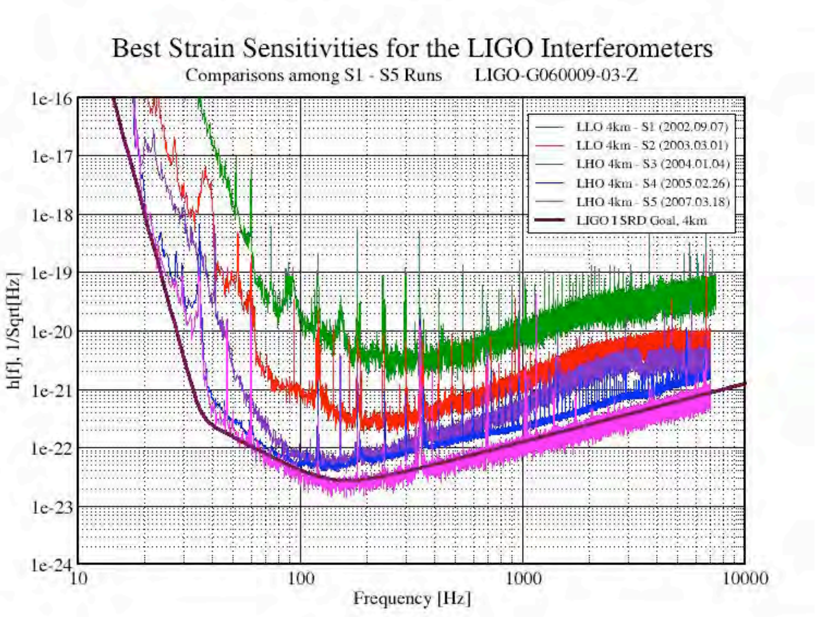
$$e_\alpha(\hat{\mathbf{\Omega}}) \;:=\; 1$$

- Time-independent overlap reduction function:

$$\gamma_{\rm iso}(f) \;=\; \int_{S^2} \mathrm{d}\hat{\mathbf{\Omega}} \left[F_1^+(\hat{\mathbf{\Omega}},t)\, F_2^+(\hat{\mathbf{\Omega}},t) \;+\; F_1^\times(\hat{\mathbf{\Omega}},t)\, F_2^\times(\hat{\mathbf{\Omega}},t)\right]\, e^{2\pi \mathrm{i} f \hat{\mathbf{\Omega}}\cdot\mathbf{\Delta x}(t)/c}$$

  - Low bandwidth (excludes detector sweet spot)





Advantage of an extended network of detectors in radiometric searches for GW

Network Performance

Band	H-L	H-L-G	H-L-V	H-L-V-G
200–300 Hz	5.79	5.43	3.44	3.04
300–400 Hz	18.57	15.37	7.92	5.88

Smallest detectable band-limited background using each of the detector networks Strain power spectrum, in units of 10^{-48} Hz^{-1}, that could be detected with 5% false alarm and 5% false dismissal rates, using one year of coincident data at design sensitivity.

Cella *et al*. (2007)

Network Performance for Anisotropic Searches

- Effective sensitivity

- Sky coverage
  - noise variance across the sky
  - better scanning

- Parameter accuracy
  - source localization

- Map making - Deconvolution, NMSE and MLR

Directed Search

$$e_\alpha(\hat{\mathbf{\Omega}}) \;:=\; \delta(\hat{\mathbf{\Omega}} - \hat{\mathbf{\Omega}}_\alpha)$$

- Direction dependent overlap reduction function:

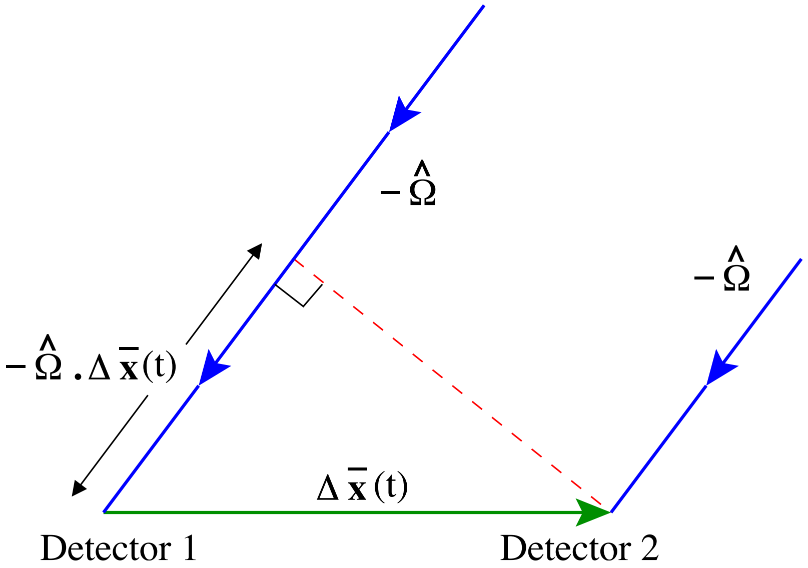
$$\gamma_{\hat{\mathbf{\Omega}}}(t,f) \;:=\; \left[F_1^+(\hat{\mathbf{\Omega}},t)\,F_2^+(\hat{\mathbf{\Omega}},t) \;+\; F_1^\times(\hat{\mathbf{\Omega}},t)\,F_2^\times(\hat{\mathbf{\Omega}},t)\right] e^{2\pi i f \hat{\mathbf{\Omega}}\cdot\mathbf{\Delta x}(t)/c}$$

Modulating factor x phase factor

- The dirty map:

$$X_{\hat{\mathbf{\Omega}}} \;\propto\; \sum_{t=0}^{T}\int_{-\infty}^{\infty} \mathrm{d}f\, \widetilde{s}_1^*(t,f)\,\widetilde{s}_2(t,f)\,\frac{H(f)\,\gamma^*_{\hat{\mathbf{\Omega}}}(t,f)}{P_1(t,|f|)\,P_2(t,|f|)}$$

  - Essentially **Earth Rotation Synthesis Imaging**



Beam / Kernel / PSF

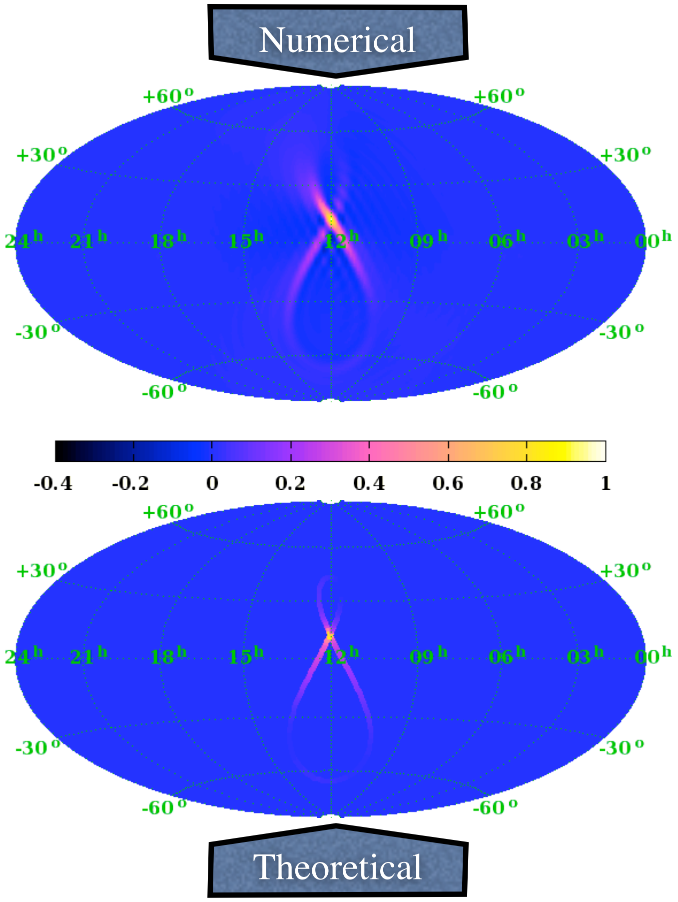
- Observed map is a convolution of Beam / Kernel / PSF and the true sky

$$\widetilde{q}_{\hat{\mathbf{\Omega}}}(t,f) \;=\; \frac{H(f)\,\gamma_{\mathcal{P}_\pm}(t,f)}{P_1(t,|f|)\,P_2(t,|f|)}$$

$$(A,B) \;:=\; \Delta T \sum_i \int_{-\infty}^{\infty} \mathrm{d}f\, P_1(t_i;|f|)\,P_2(t_i;|f|)\,\widetilde{A}^*(t_i;f)\,\widetilde{B}(t_i;f)$$

$$X_{\hat{\mathbf{\Omega}}} \;=\; \left(q_{\hat{\mathbf{\Omega}}}, \frac{\widetilde{s}_1^*(t,f)\,\widetilde{s}_2(t,f)}{P_1(t,|f|)\,P_2(t,|f|)}\right)$$

$$\boxed{B(\hat{\mathbf{\Omega}},\hat{\mathbf{\Omega}}') \;=\; (q_{\hat{\mathbf{\Omega}}}, q_{\hat{\mathbf{\Omega}}'})/\|q_{\hat{\mathbf{\Omega}}}\|^2}$$

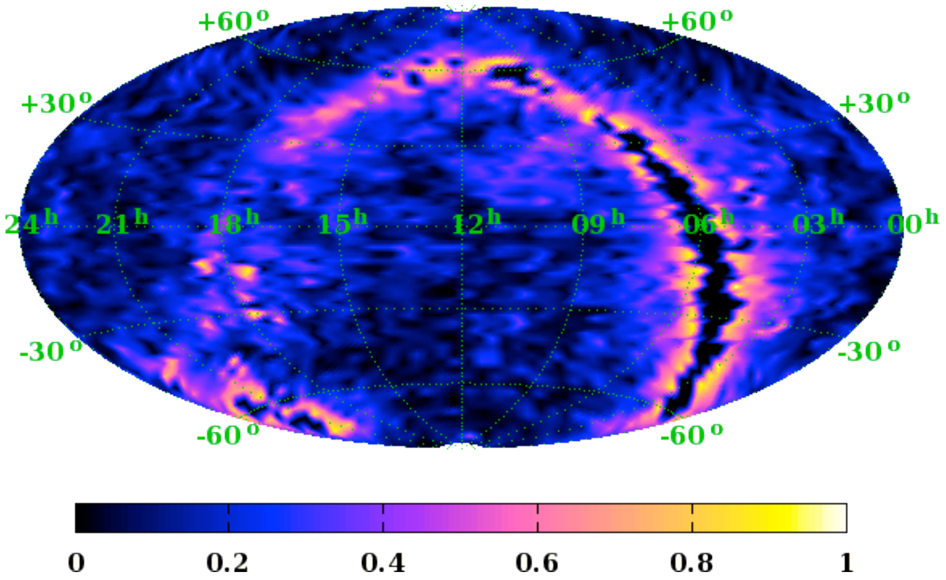


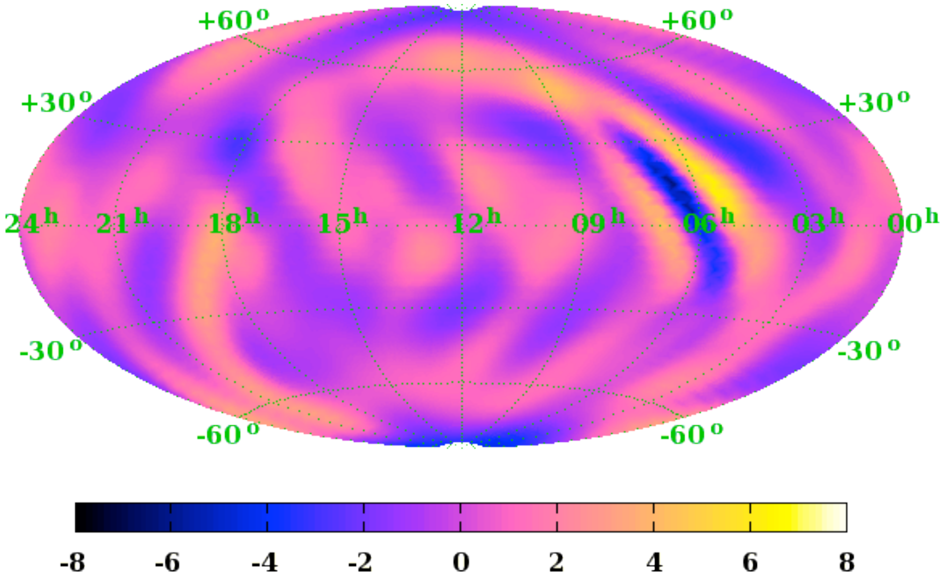
  - Stationary Phase Approximation provides a nice theoretical model

Example of Directed Radiometer Deconvolution

- Toy Multi-declination Source

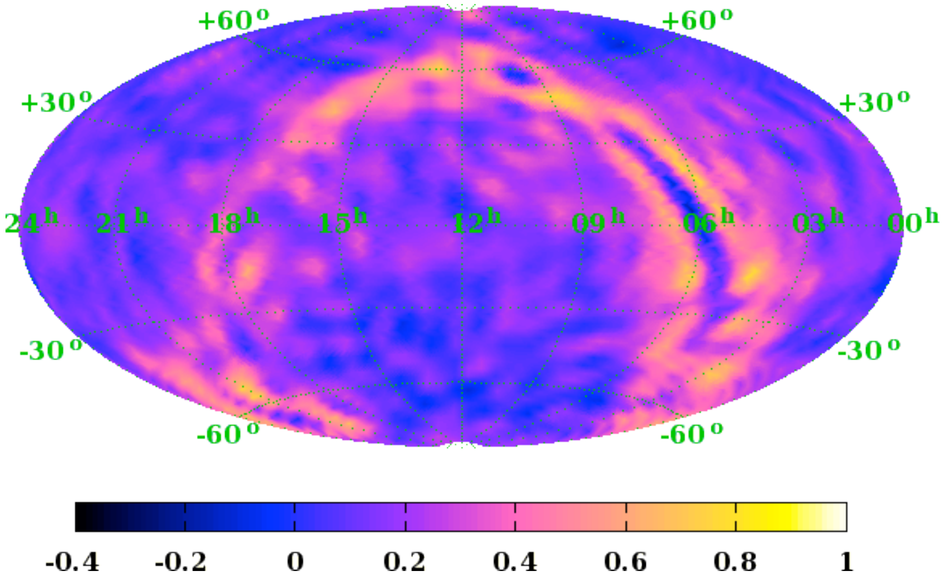
Injected

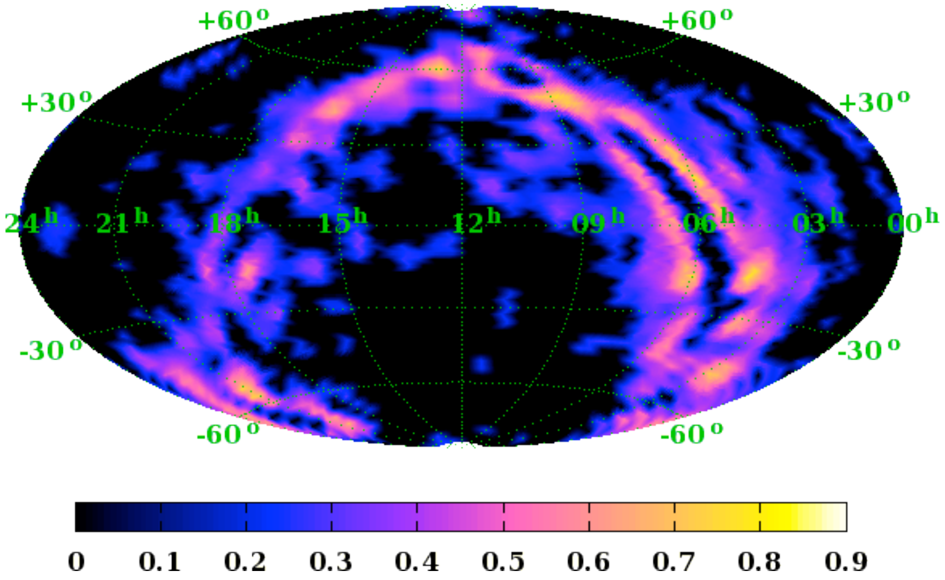




Dirty

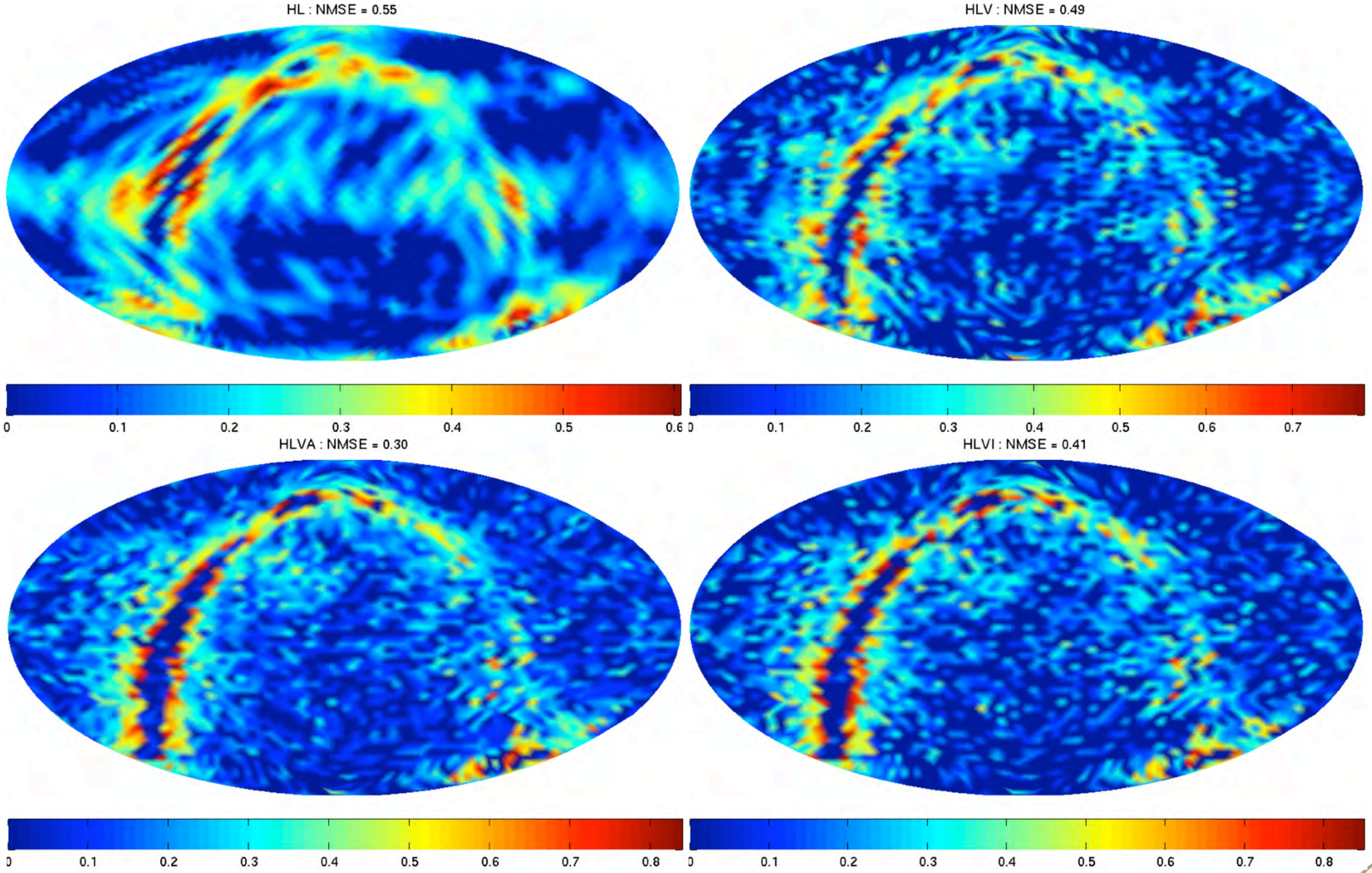
Clean





Masked

Network Deconvolution Performance



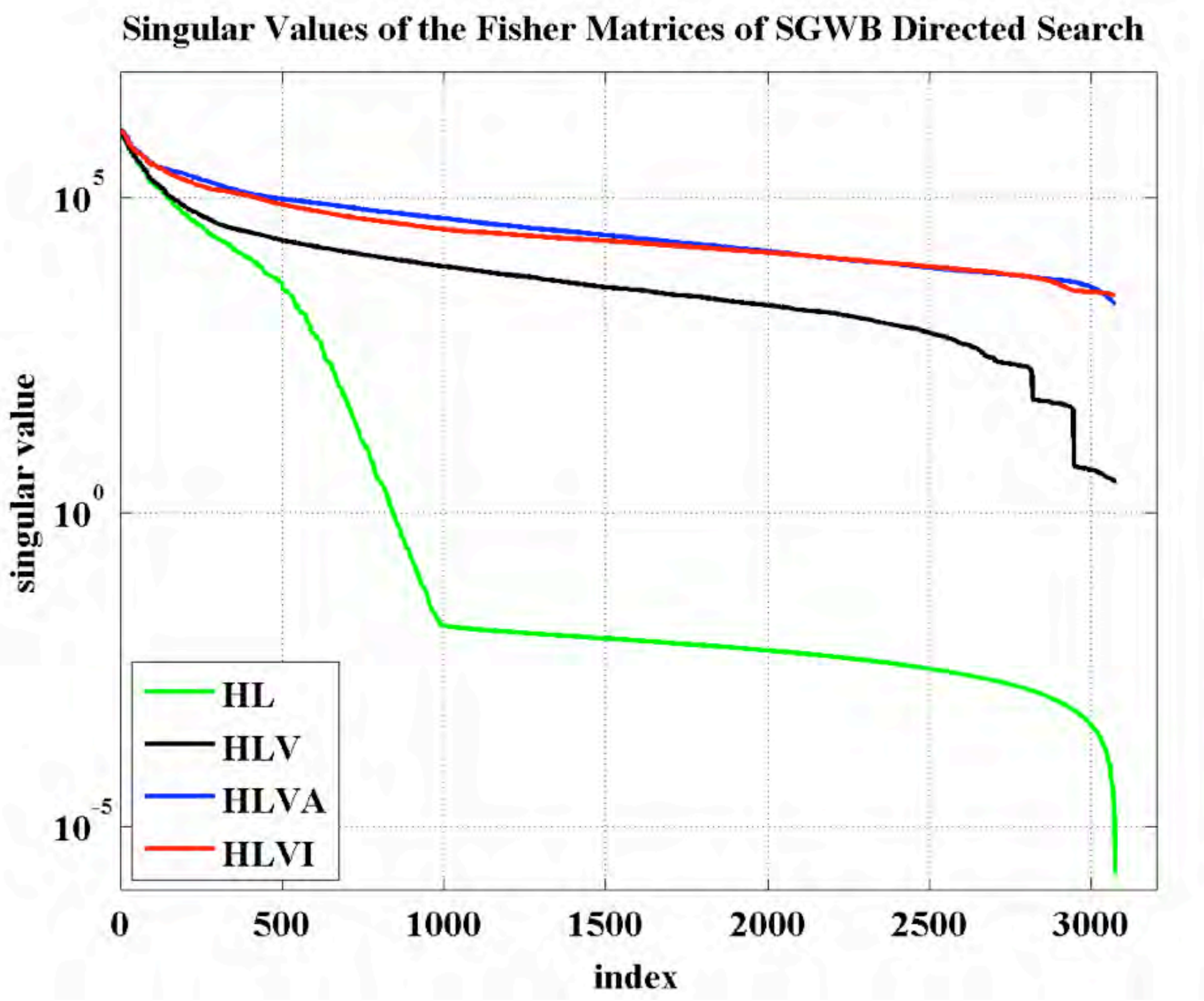
HL
NMSE:
0.55

HLV
NMSE:
0.49

HLVA
NMSE:
0.30

HLVI
NMSE:
0.41

Singular Values of a Network “Fisher Matrix”



Spherical Harmonic (SpH) Basis

$$e_{lm}(\hat{\mathbf{\Omega}}) \;:=\; Y_{lm}(\hat{\mathbf{\Omega}})$$

- Harmonic space overlap reduction function

$$\gamma^{I}_{lm}(f,t) \;:=\; \sum_{A=+,\times} \int_{S^2} \mathrm{d}\hat{\mathbf{\Omega}}\, F^{A}_{I_1}(\hat{\mathbf{\Omega}},t)\, F^{A}_{I_2}(\hat{\mathbf{\Omega}},t)\, e^{2\pi \mathrm{i} f \hat{\mathbf{\Omega}}\cdot\mathbf{\Delta x}(t)/c}\, Y_{lm}(\hat{\mathbf{\Omega}})$$

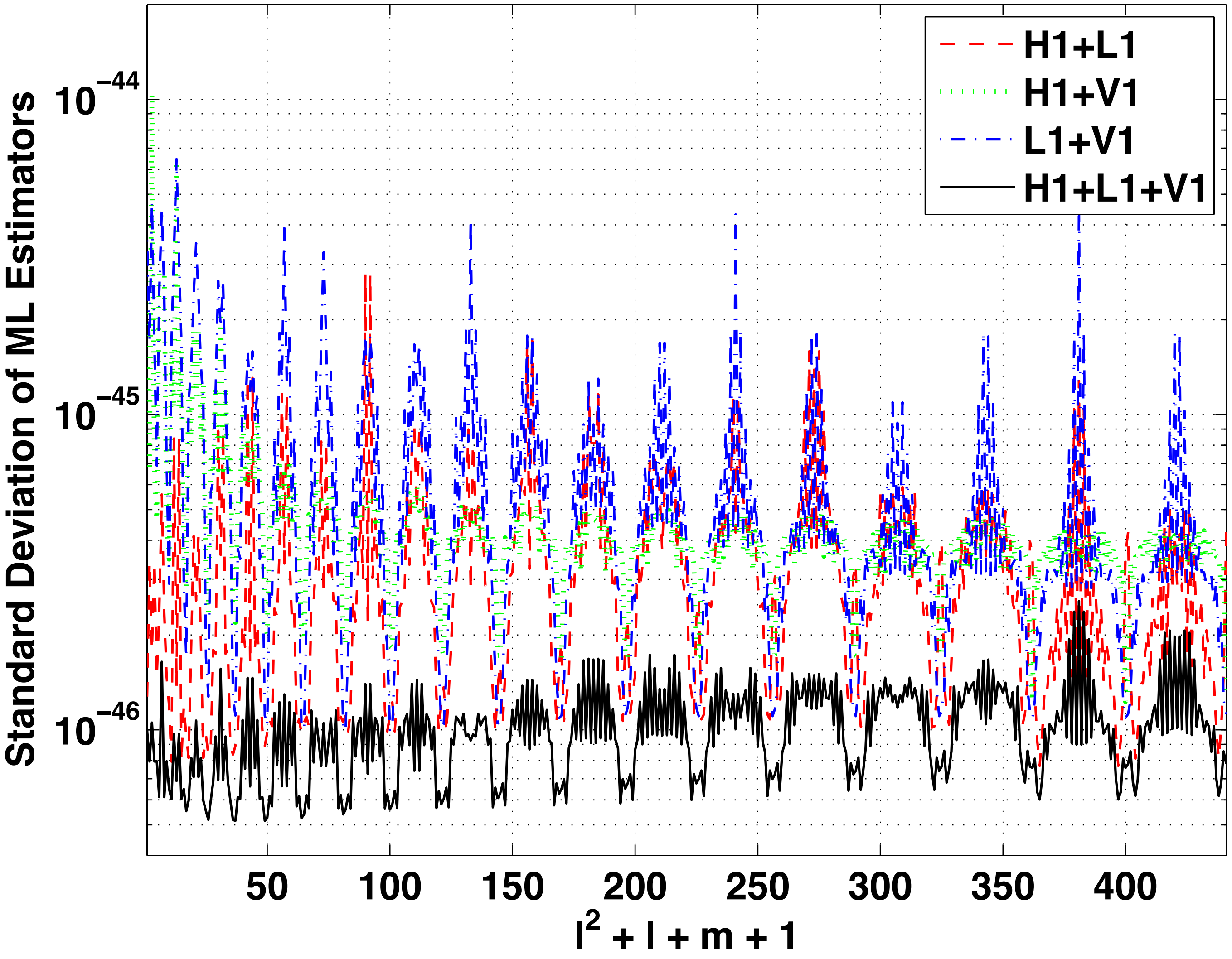
  - analytically computed
  - has the nice azimuthal symmetry: $\gamma^{I}_{lm}(f,t) \;=\; \gamma^{I}_{lm}(f,0)\, \exp \dfrac{t_{\mathrm{sidereal}}}{1 \text{ sidereal day}}$

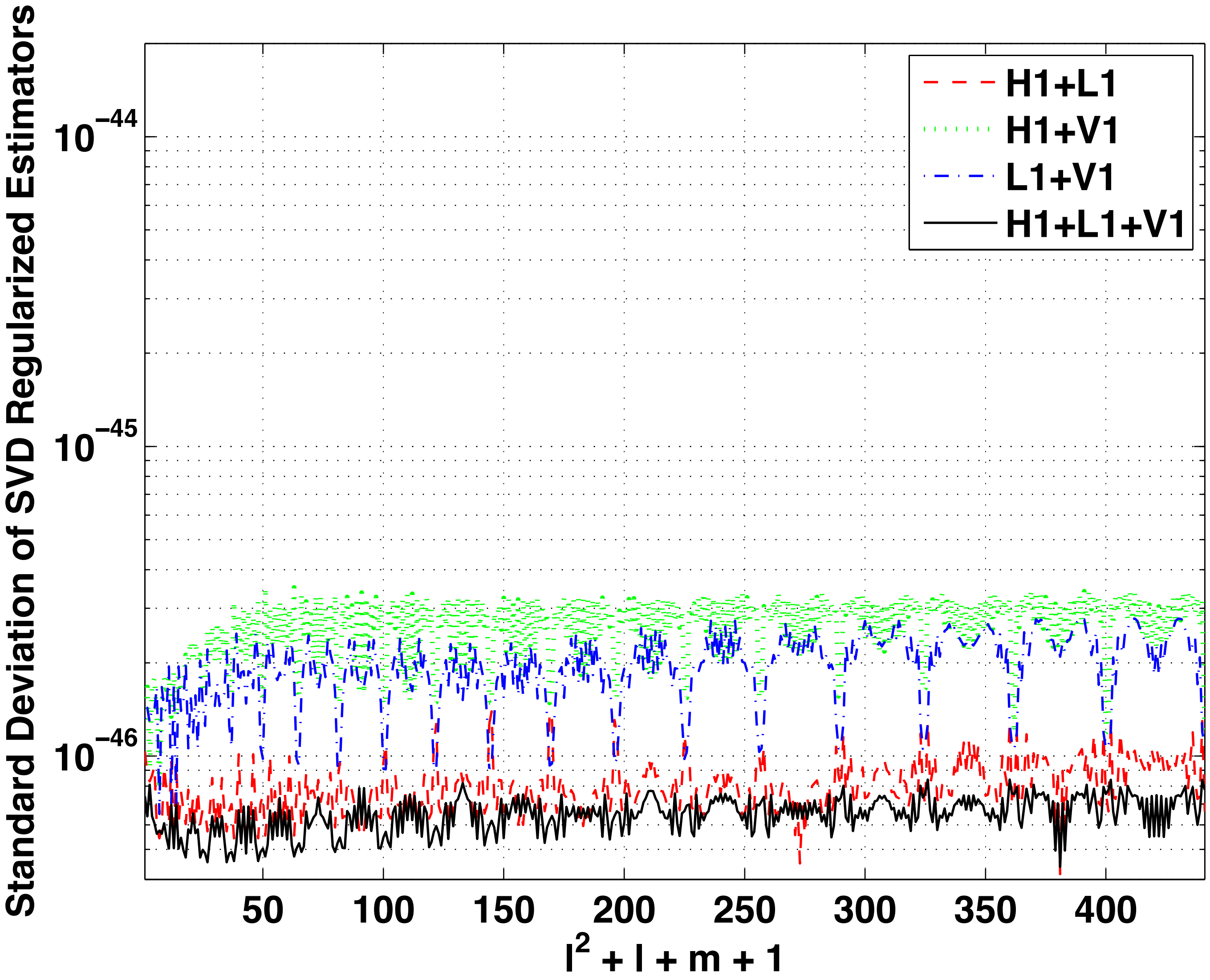
- Why Spherical Harmonic basis?
  - easy to impose natural physical cutoffs
  - easy to get the noise covariance matrix of the estimated map

e.g., $l_{max} < 10$ cut off can not be applied in the pixel basis. Though high res cutoffs, like $l_{max} < 1000$, are still possible in pixel basis

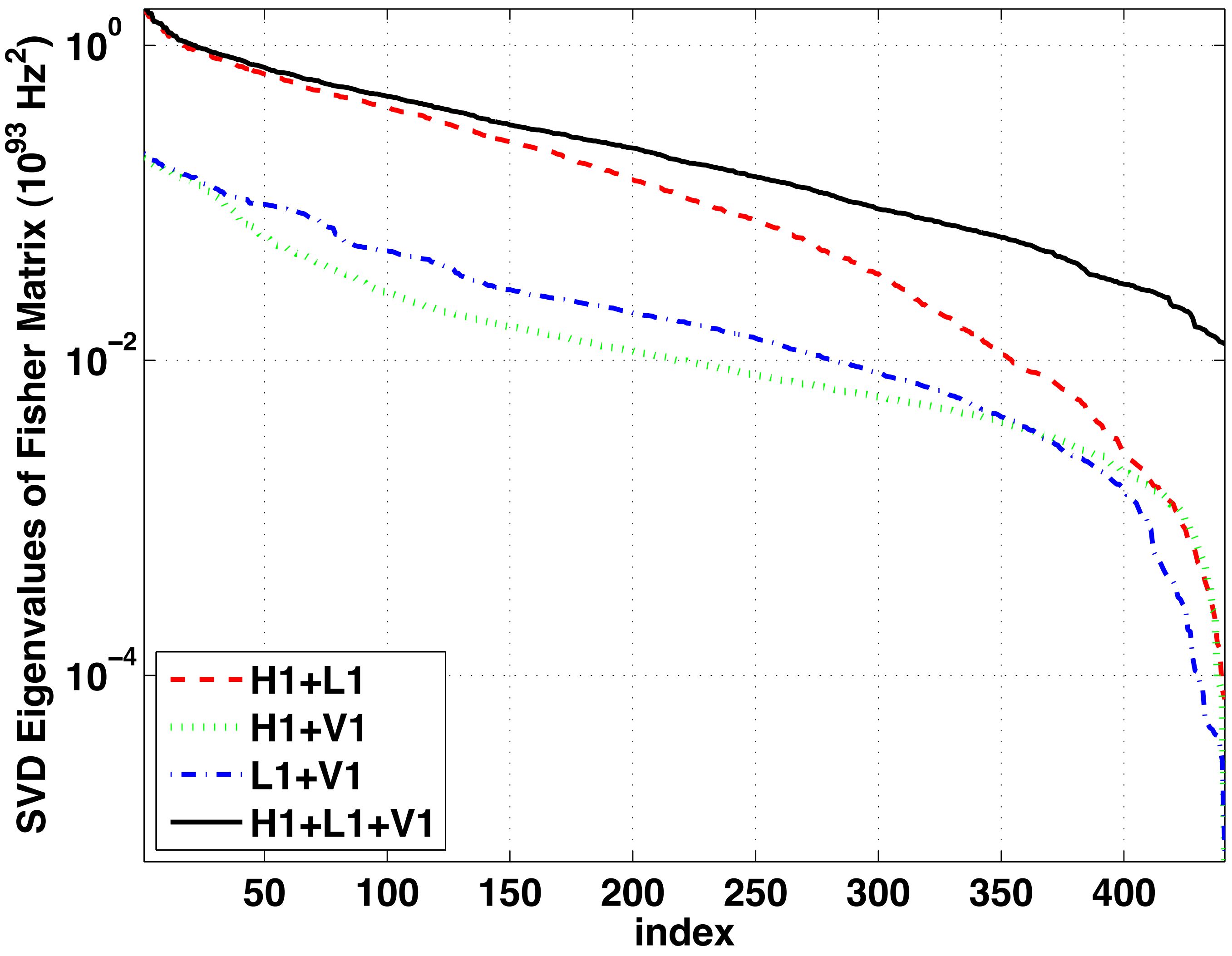
Advantage of a Network

- higher sensitivity and more uniform sky coverage





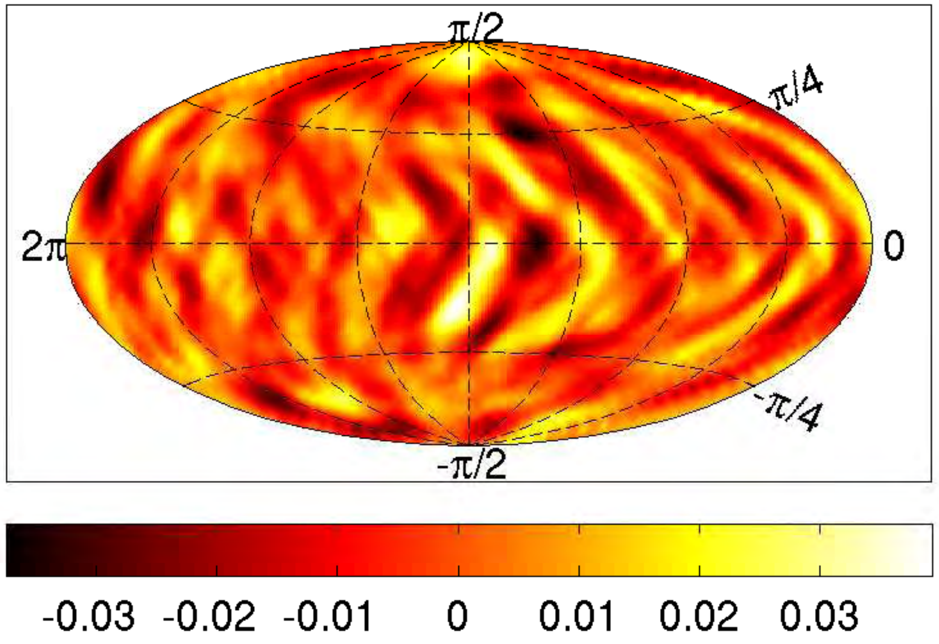
Singular Values of a Network SpH “Fisher Matrix”

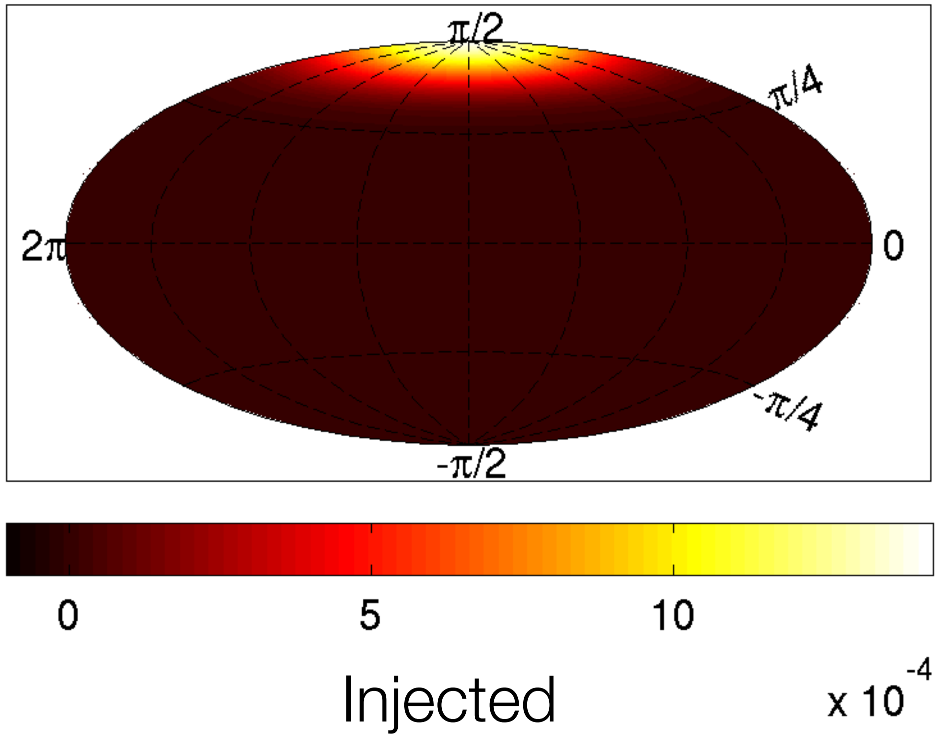


Search using Max Likelihood Ratio (MLR) Statistic

- Result of a weak injection

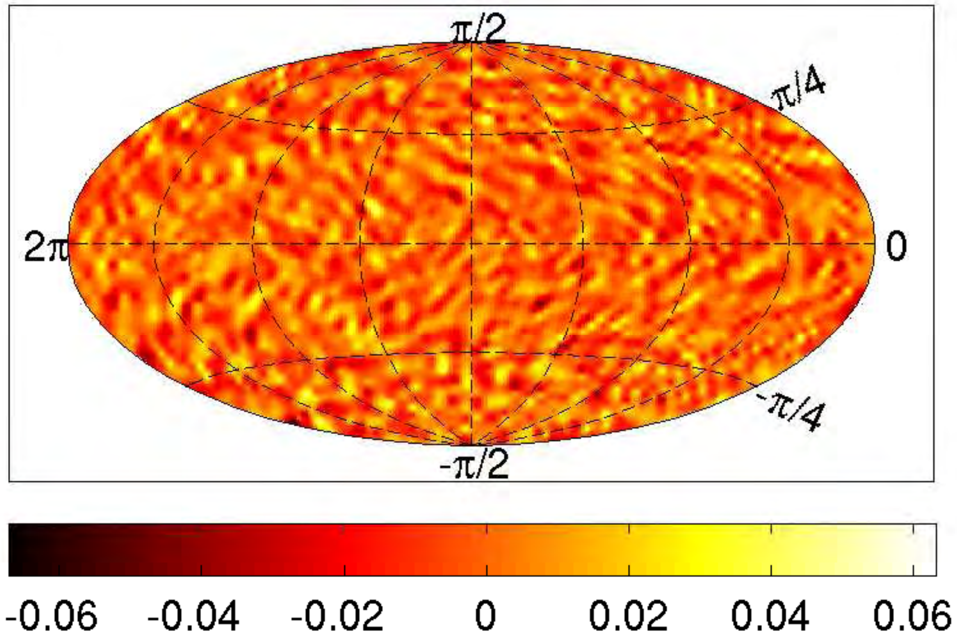
Dirty Map





Injected

$$\lambda = \frac{S_k \hat{\mathcal{P}}^k}{\sqrt{\hat{\mathcal{P}}^q \mathcal{B}_{qr} \hat{\mathcal{P}}^r}}$$

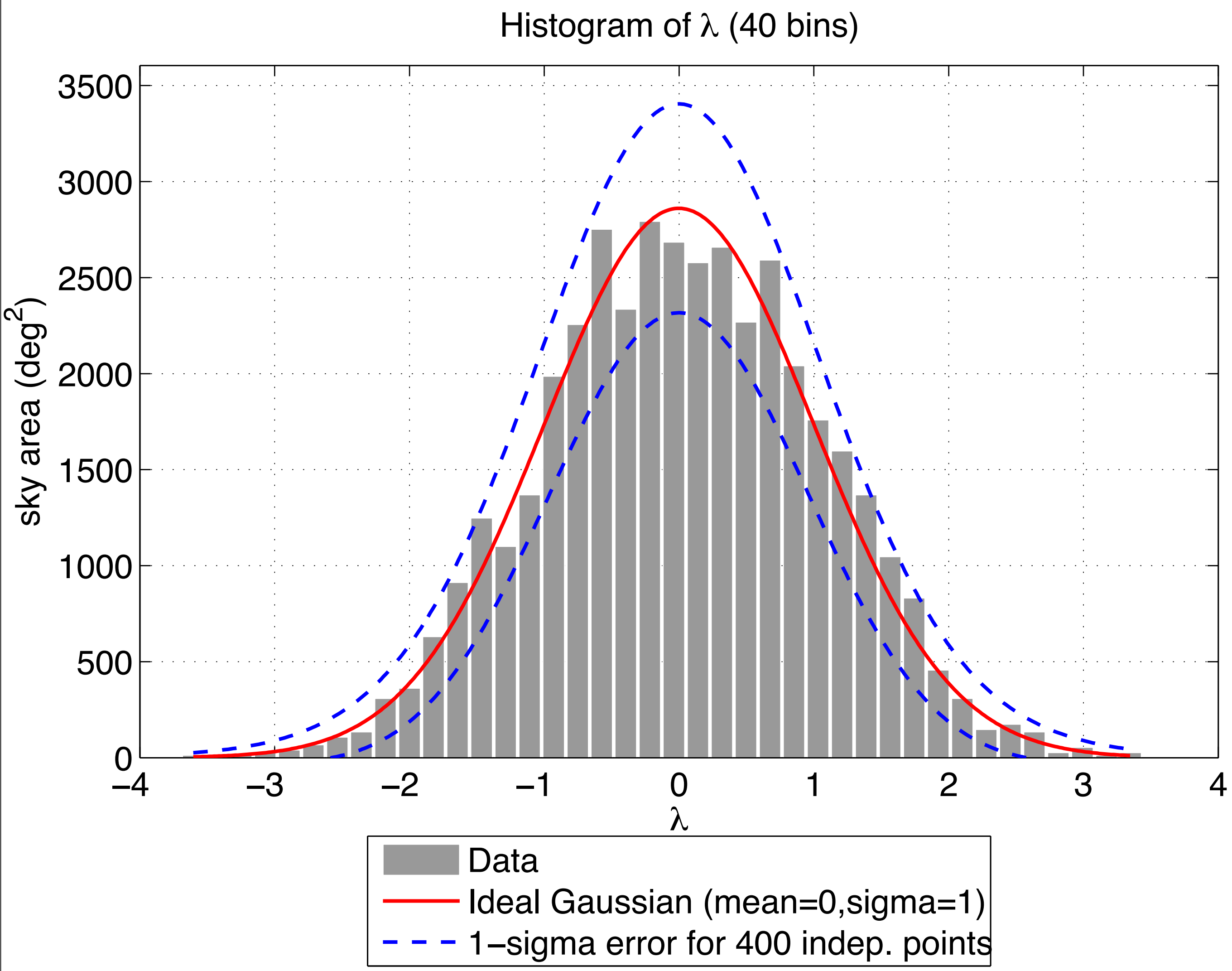


Clean Map

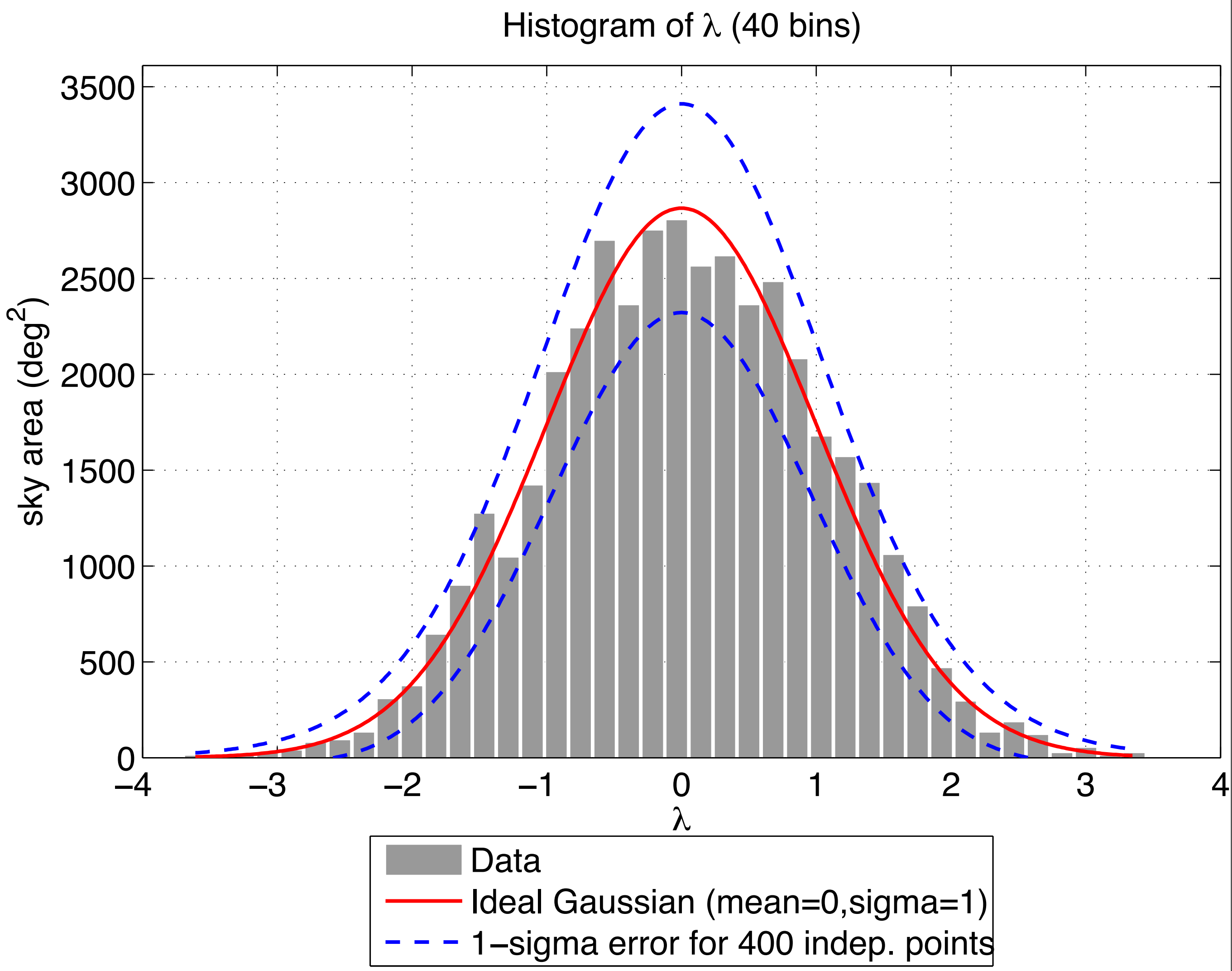
Search using Max Likelihood Ratio (MLR) Statistic

- Distribution of radiometer map

Noise Only

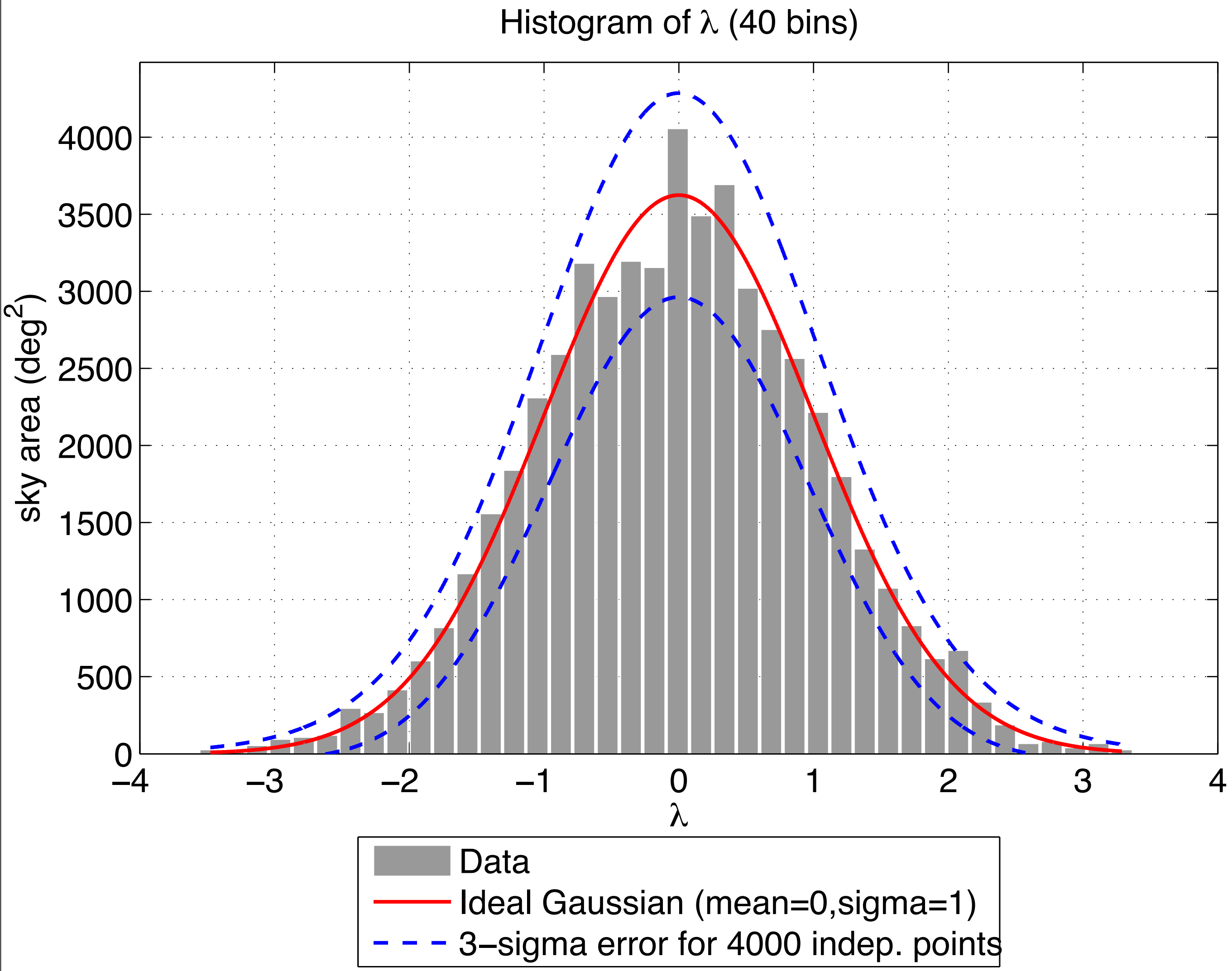


Weak Injection

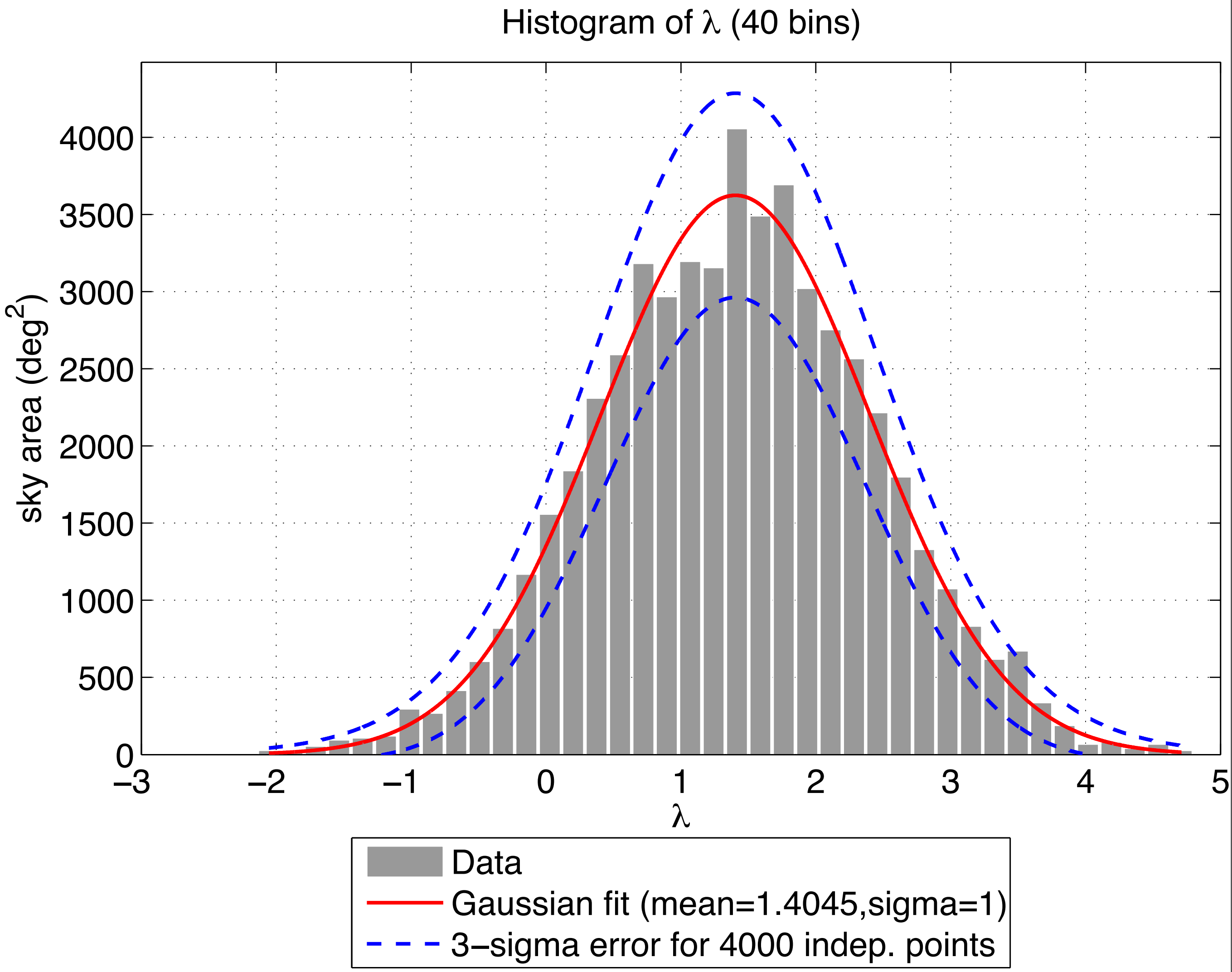


Search using Max Likelihood Ratio (MLR) Statistic

- Distribution of MLR statistic for 4000 MC realizations



Noise Only



Weak Injection

MLR Statistic Search Network Performance

- MLR statistic obtained from dirty (λ) and clean (λ_c) maps
  - HLV network performance is ~15% better than HL baseline

Noise Only

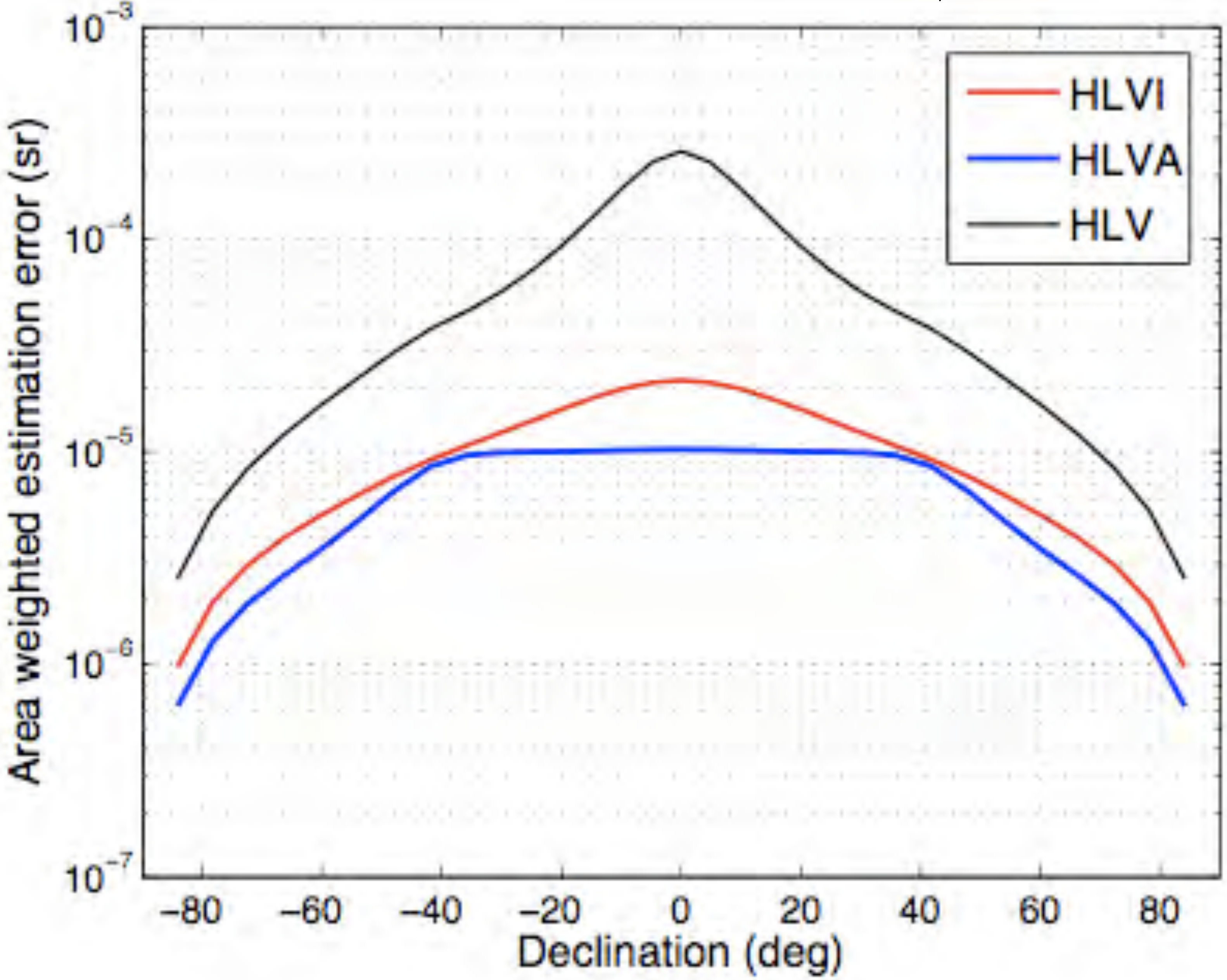
Baseline	λ	λ_c
H1L1	0.0512	0.0433
L1V1	-0.1549	-0.1542
H1V1	0.1105	0.1120
H1L1V1	0.0208	0.0149

Strong Injection

Baseline	λ	λ_c
H1L1	78.5555	78.3271
L1V1	35.9004	35.8940
H1V1	31.5717	31.5662
H1L1V1	91.9594	91.7600

Source Localization Error with a Network

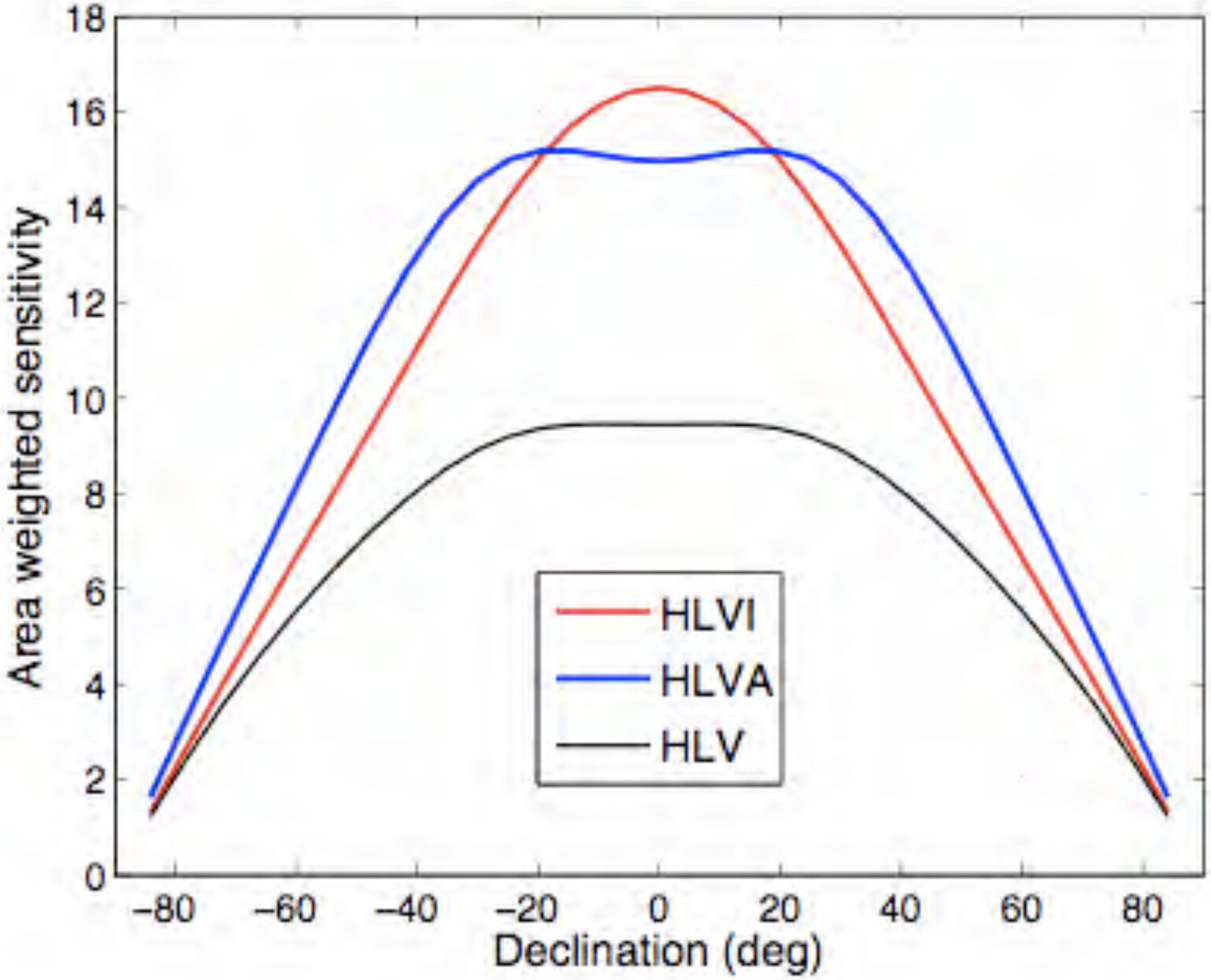
$$\cos(\mathrm{dec})\, 2\pi \sqrt{\langle(\Delta\cos\theta)^2\rangle_{\hat{\Omega}}\,\langle(\Delta\phi)^2\rangle_{\hat{\Omega}} - \langle(\Delta\cos\theta)(\Delta\phi)\rangle_{\hat{\Omega}}^2}$$



Advantage of an extended network of detectors in radiometric searches for GW

Network “Sensitivity”

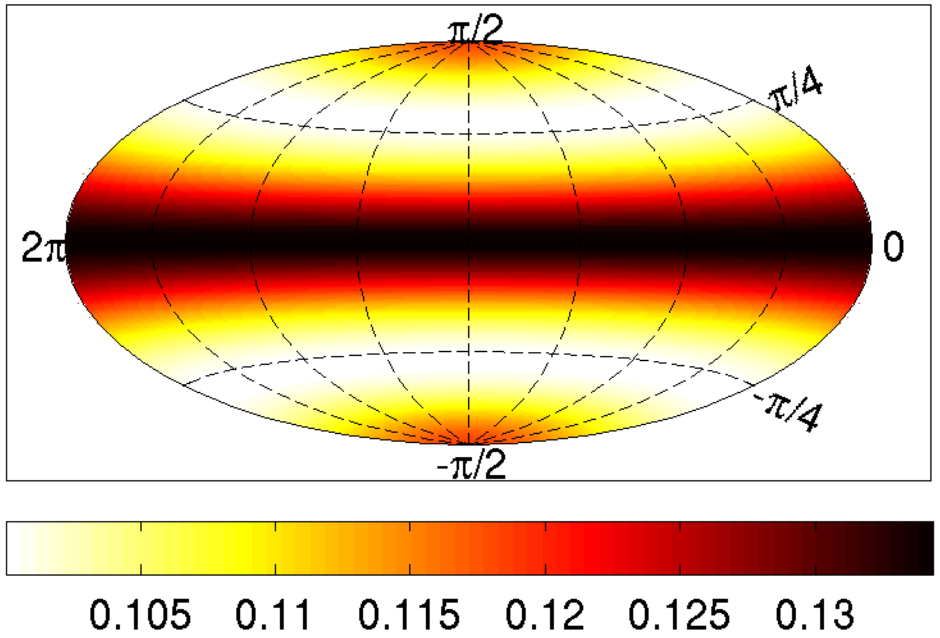
$$\cos(\mathrm{dec})\,\|q_{\hat{\boldsymbol{\Omega}}}\|$$



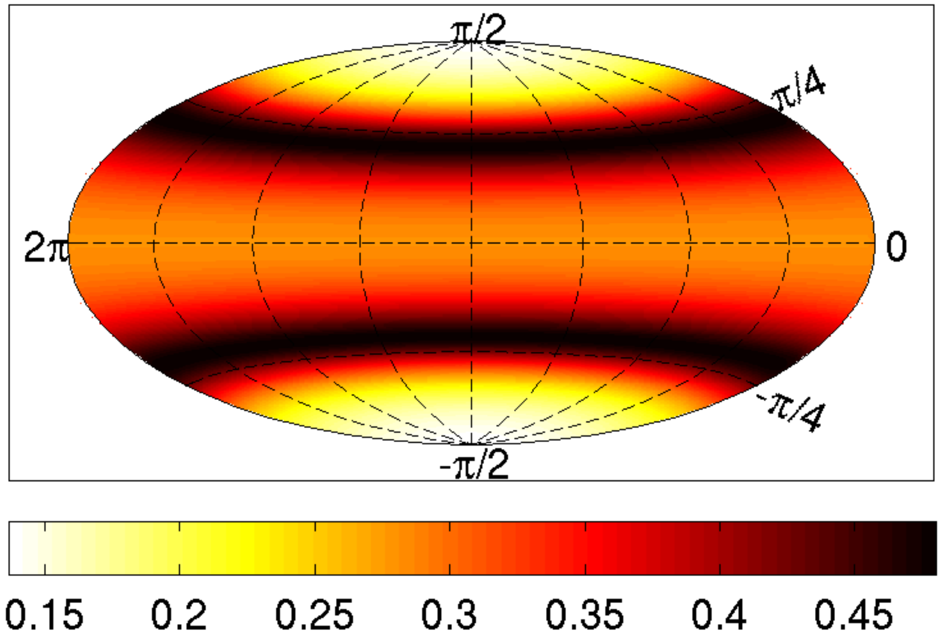
Advantage of an extended network of detectors in radiometric searches for GW

Network Sky Coverage: Standard Deviation

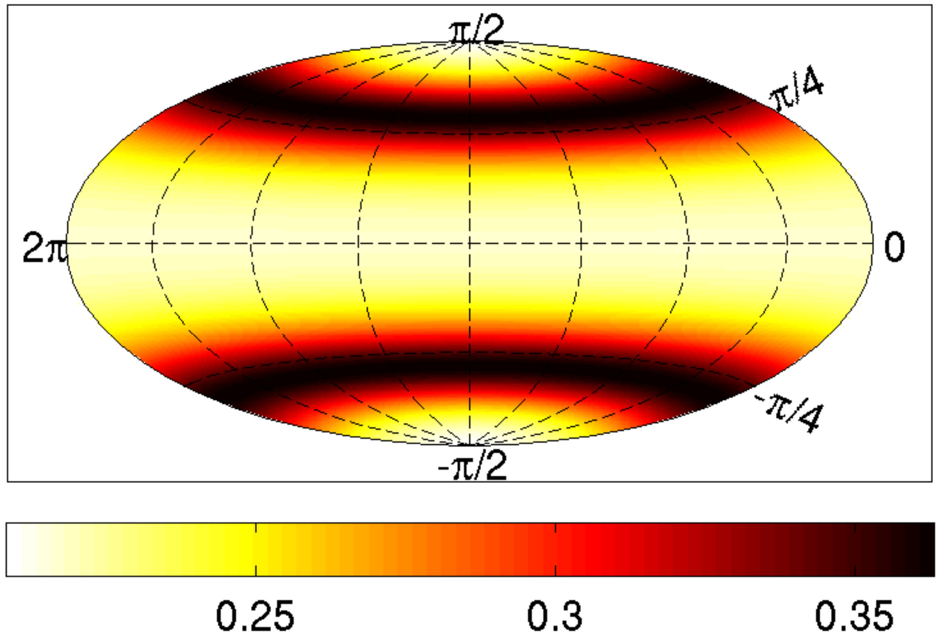
HL



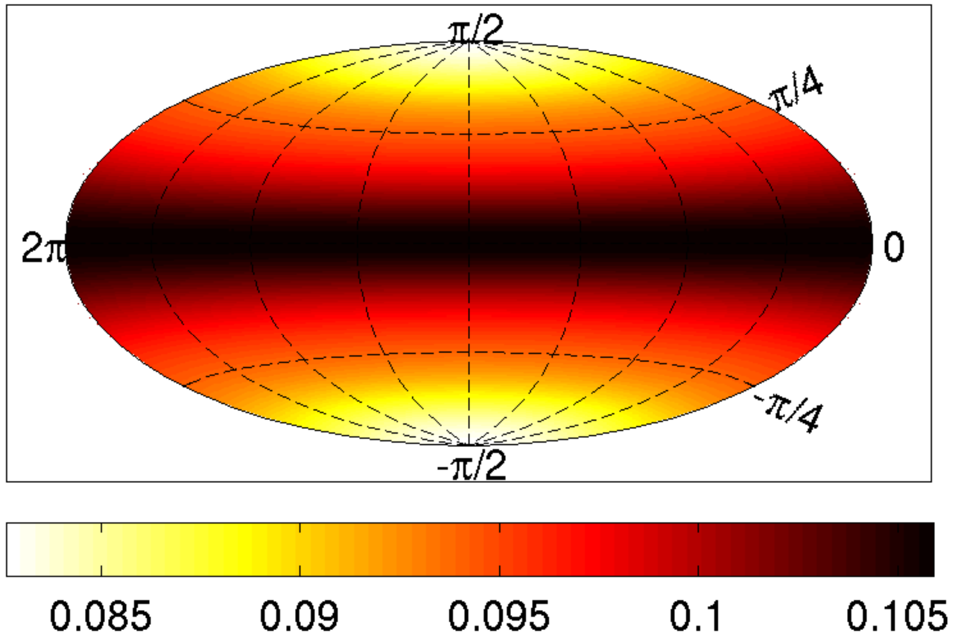
HV



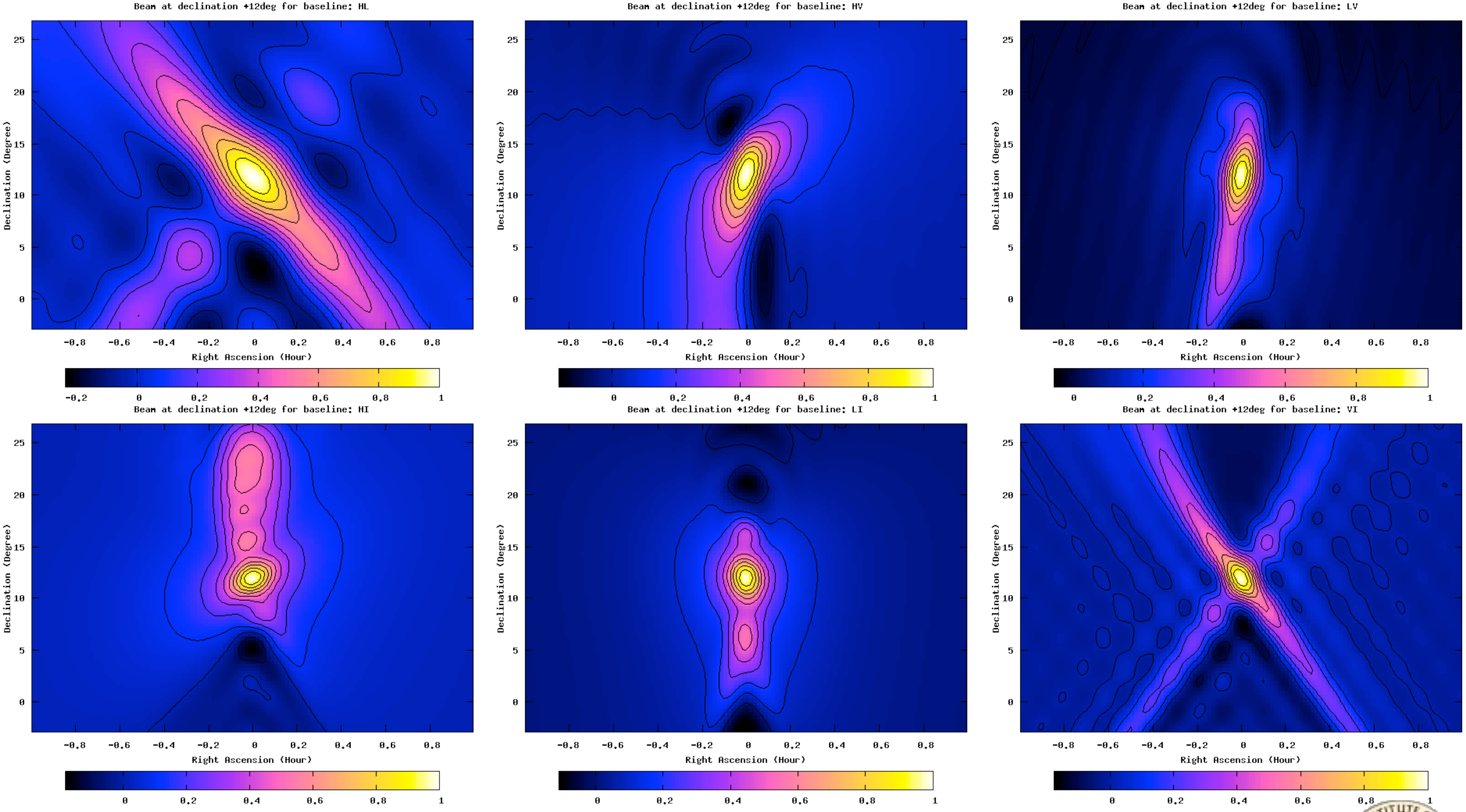
LV



HLV



Network Sky Coverage: Scanning

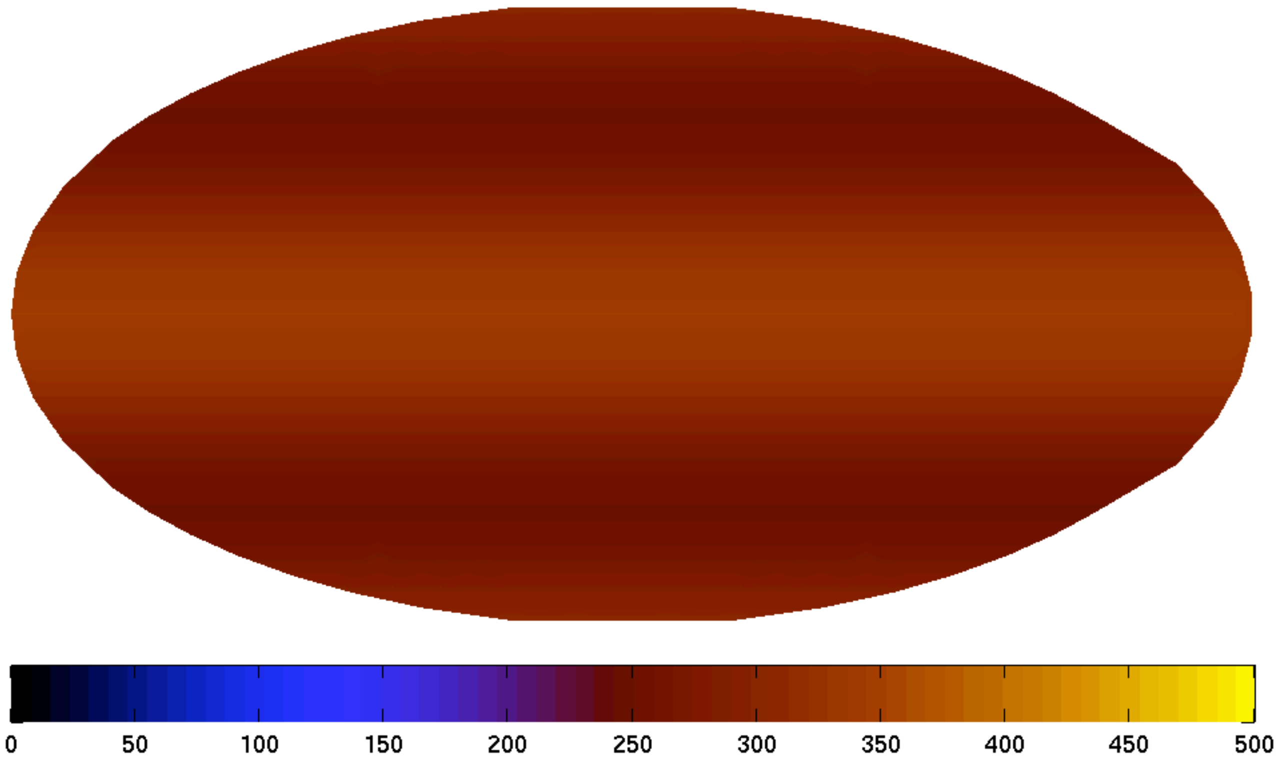


Advantage of an extended network of detectors in radiometric searches for GW

Finding Optimal Detector Orientation

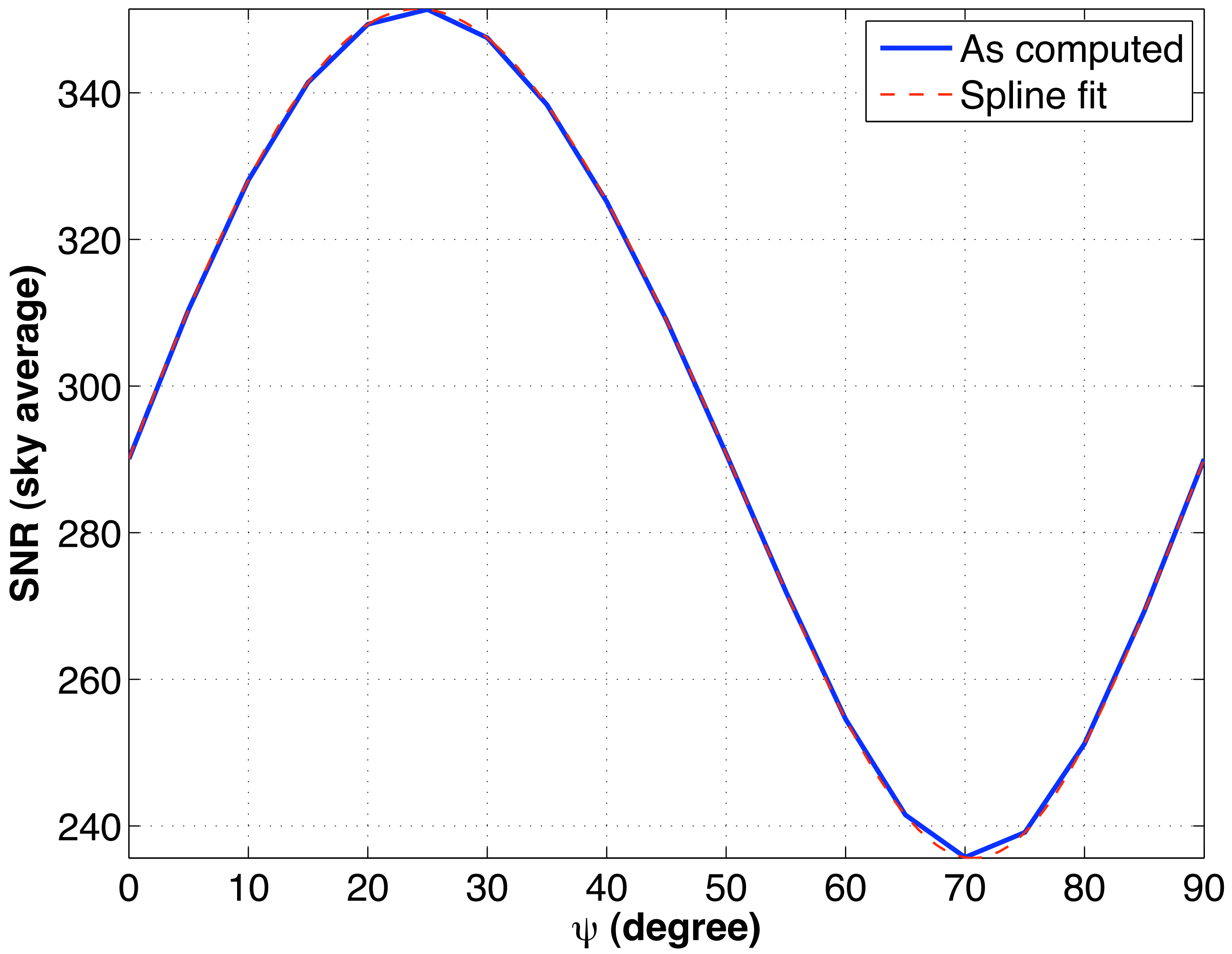
- Rotate detector and plot sky SNR map

$$\mathrm{SNR}_{\hat{\mathbf{\Omega}},\alpha} = \frac{2\,\mathcal{P}_{\hat{\mathbf{\Omega}}}}{100^{\alpha}}\sqrt{\sum_{i=1}^{n}\int_{-\infty}^{\infty}\mathrm{d}f\,\frac{\mathrm{f}^{2\alpha}\,|\gamma_{\hat{\mathbf{\Omega}}}(\mathrm{t_i},\mathrm{f})|^2}{\mathrm{P}_1(\mathrm{t_i};|\mathrm{f}|)\,\mathrm{P}_2(\mathrm{t_i};|\mathrm{f}|)}}$$



Finding Optimal Detector Orientation

- Plot orientation vs sky averaged SNR



Conclusions

- A general maximum likelihood (ML) framework to search for SGWB using the radiometer algorithm is useful for studying the performance of a network

- We have used different figures of merits to compare the performances of the network with its individual baselines

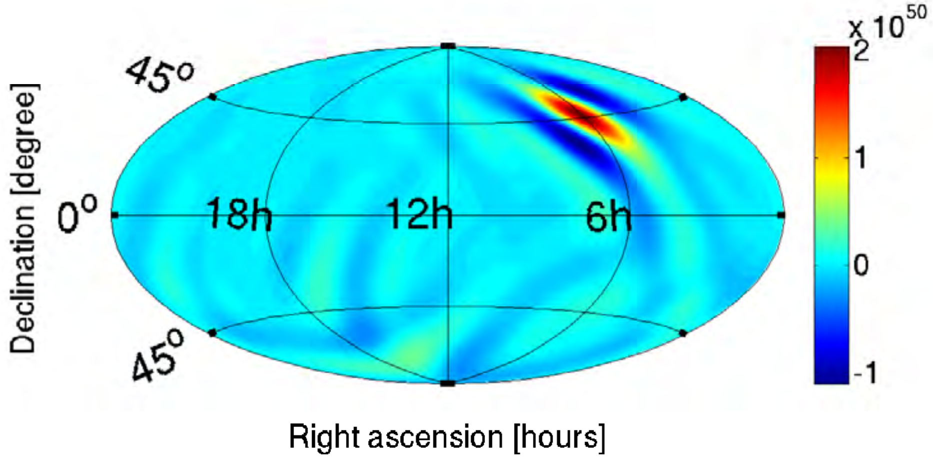
- Some results have been derived for a detector in India/Australia, a more organized study is necessary to complete this exercise

- Radiometer analysis has important applications (e.g., SGWB, pulsar searches)
  - also useful to obtain quick results and may provide insights on where to push the analysis to extract more science from a network

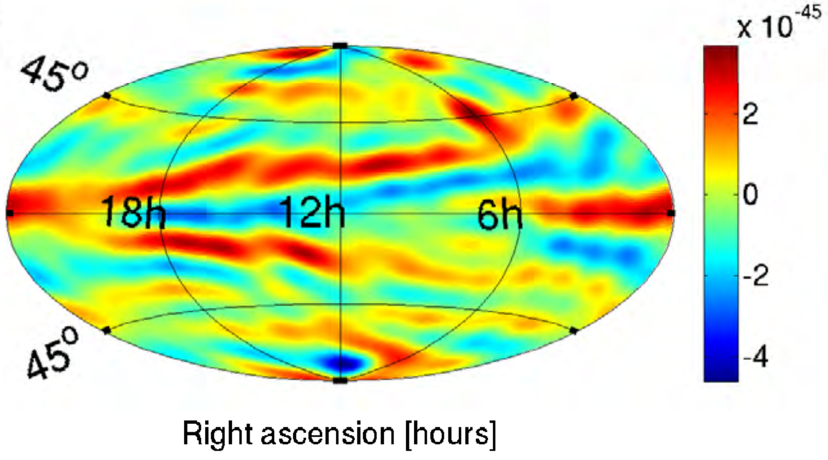
Thank You!

Spherical Harmonic Basis Implementation

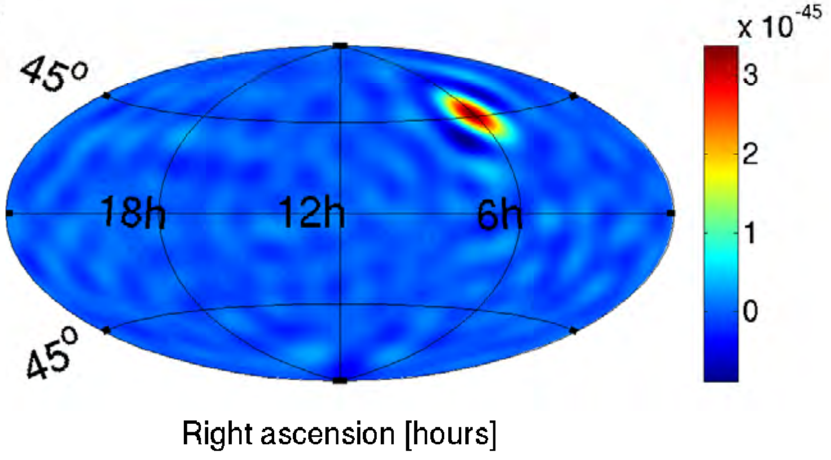
- Point source recovered nicely



Dirty Map

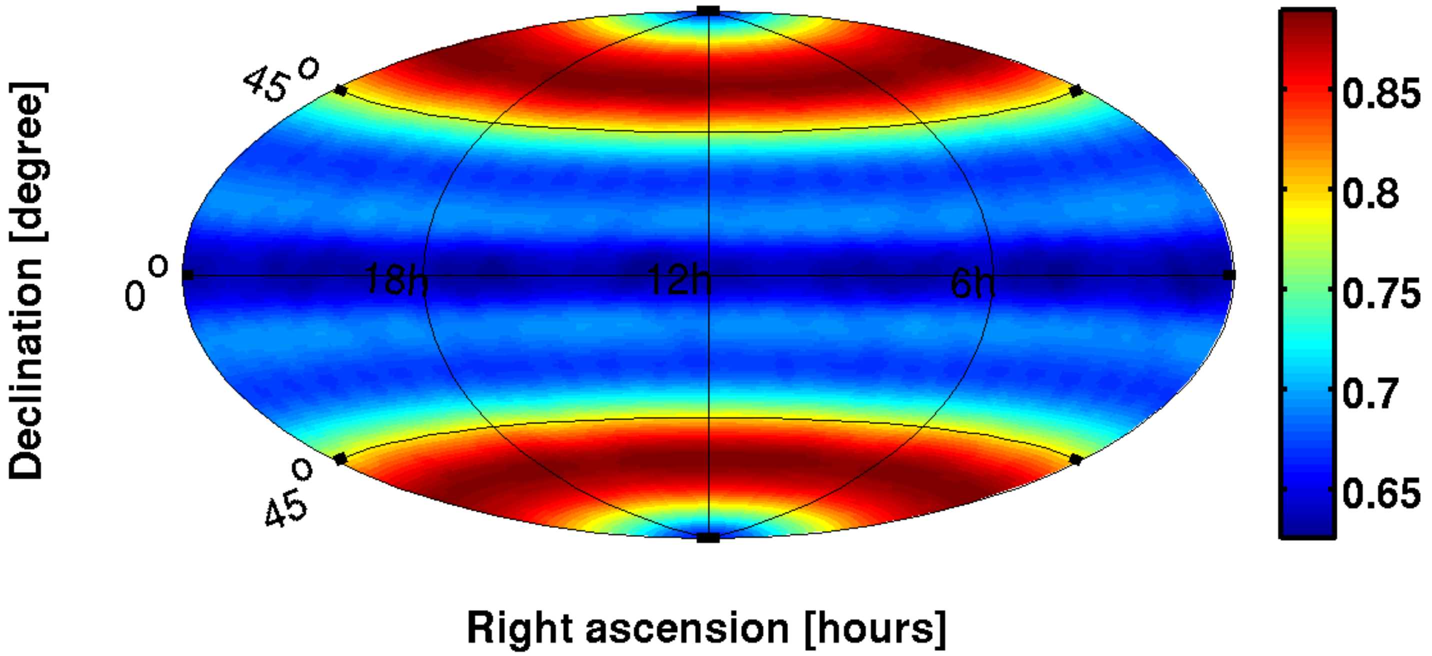


Clean Map w/o regularization



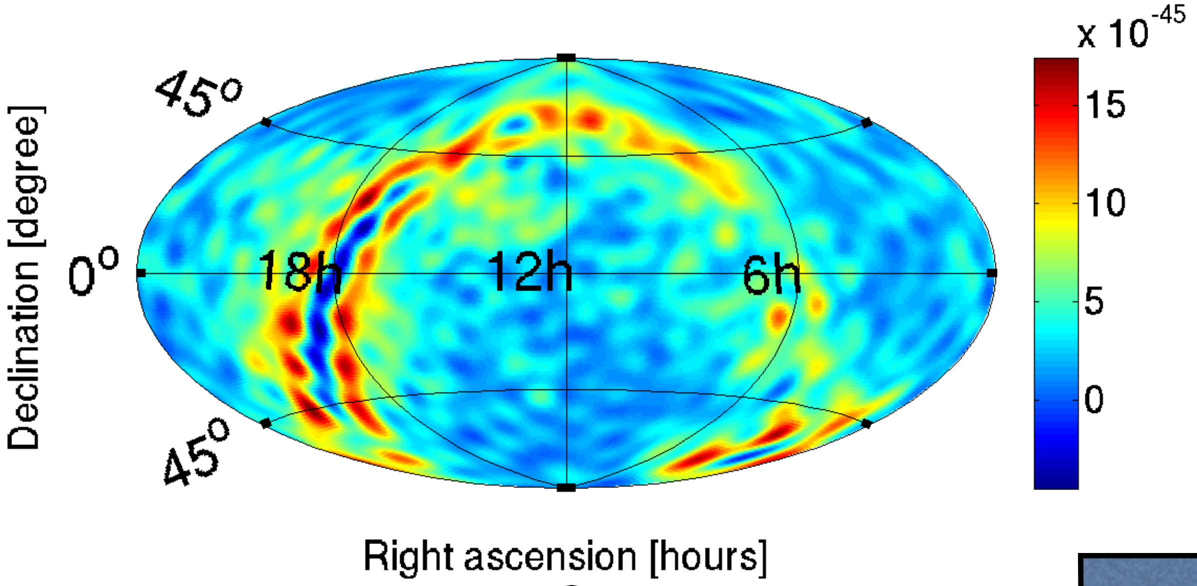
Clean Map w/ SVD regularization

- But, SVD introduces bias



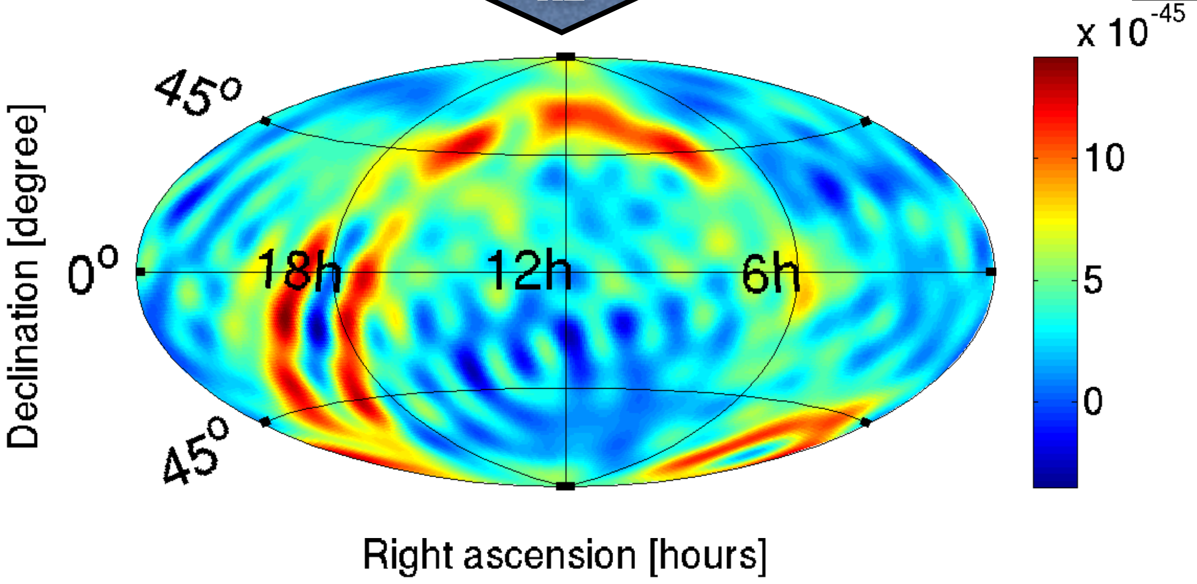
Spherical Harmonic Basis Implementation

- Toy multi-declination injection

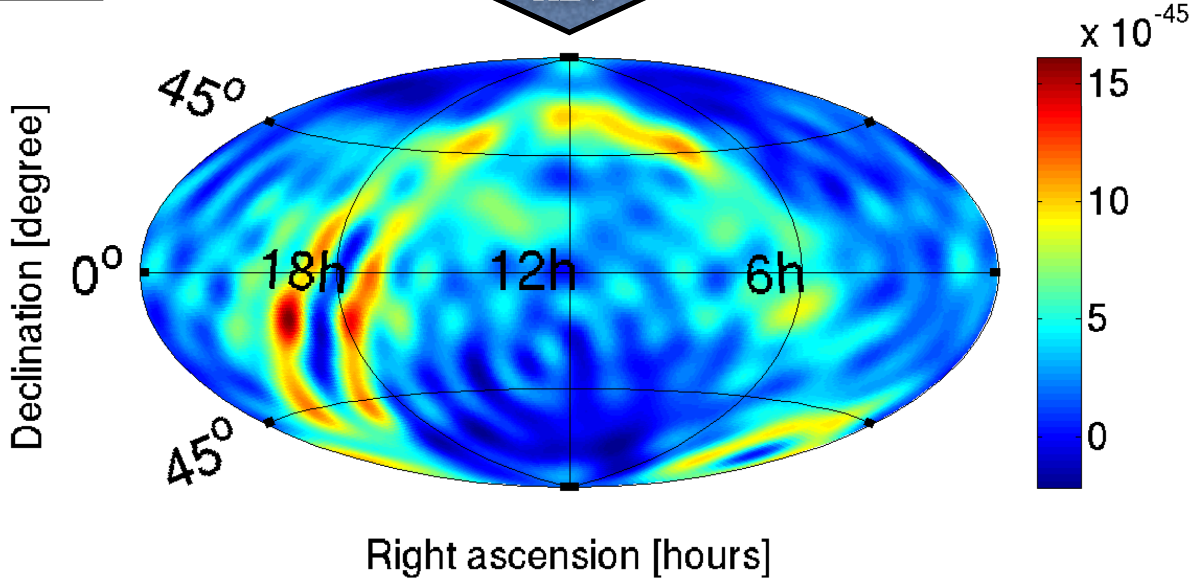


Injected

Clean HL



Clean HLV



Stochastic Gravitational Wave Background (SGWB)

- Unresolved astrophysical or cosmological sources
  - popcorn or continuous
- Carry information not accessible in electro-magnetic astronomy
  - astrophysical sources
    - information on the anisotropic local universe
  - primordial cosmological background (CGWB)
    - direct probe of **inflation**
- Why here? A quick & comprehensive way to test network sensitivity/coverage

Quantities to Measure

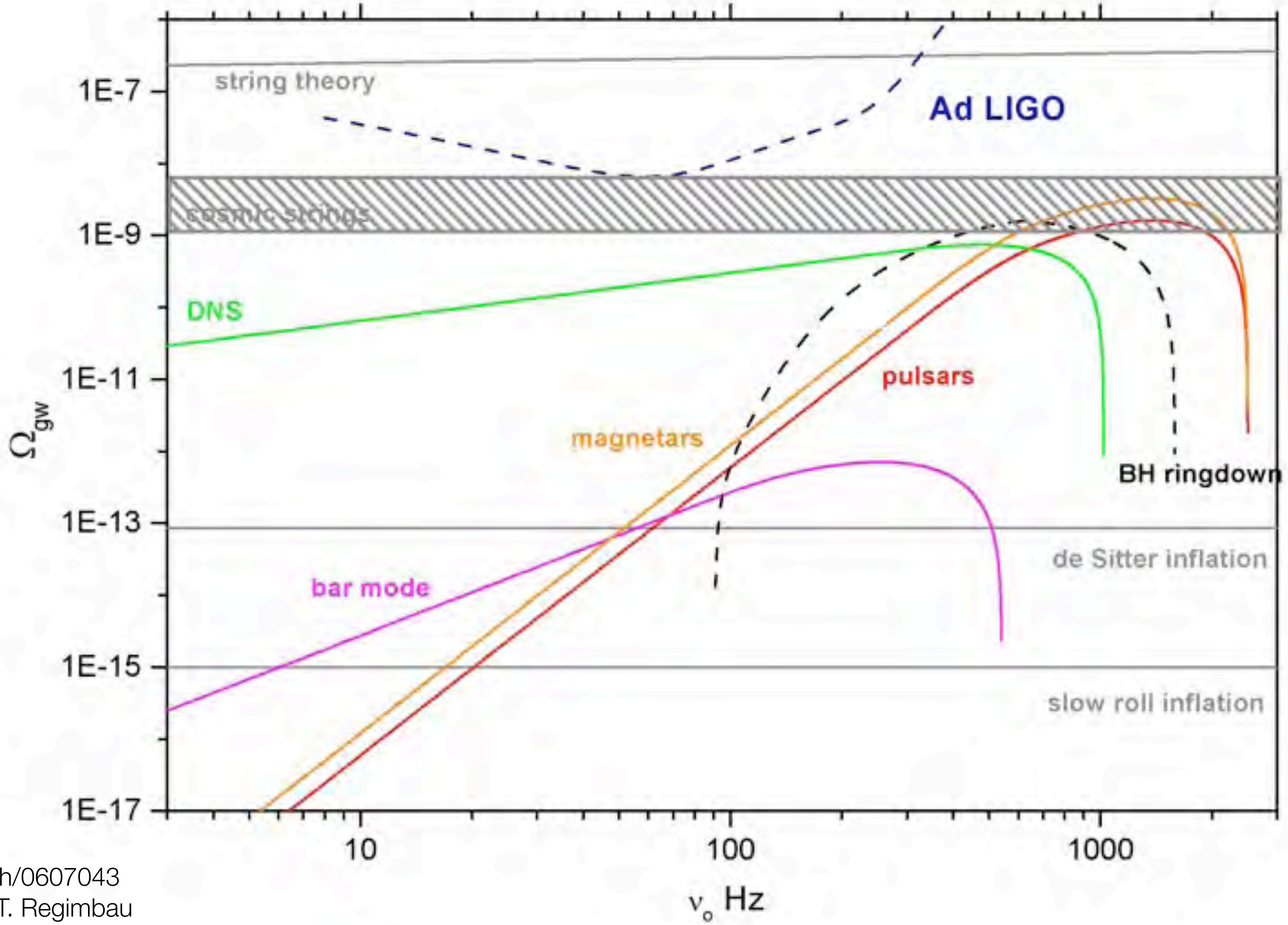
- Definition: $\langle \tilde{h}_A(f,\hat{\mathbf{\Omega}})\tilde{h}_{A'}(f',\hat{\mathbf{\Omega}}')\rangle \;=\; \delta_{AA'}\,\delta(f-f')\,\delta^2(\hat{\mathbf{\Omega}},\hat{\mathbf{\Omega}}')\,\mathcal{P}_A(\hat{\mathbf{\Omega}})\,H(f); \quad A, A' = +, \times$

- SGWB spectrum: $$\boxed{\Omega_{\mathrm{GW}}(f) \;=\; \frac{1}{\rho_{\mathrm{crit}}}\,\frac{\mathrm{d}\rho_{\mathrm{GW}}(f)}{\mathrm{d}\ln f}}$$
  - energy density per unit frequency interval in the units of critical density of the universe that is needed to make it flat

- **Specific intensity** of Gravitational Waves (GW):
  - GW flux incident normally per unit solid angle

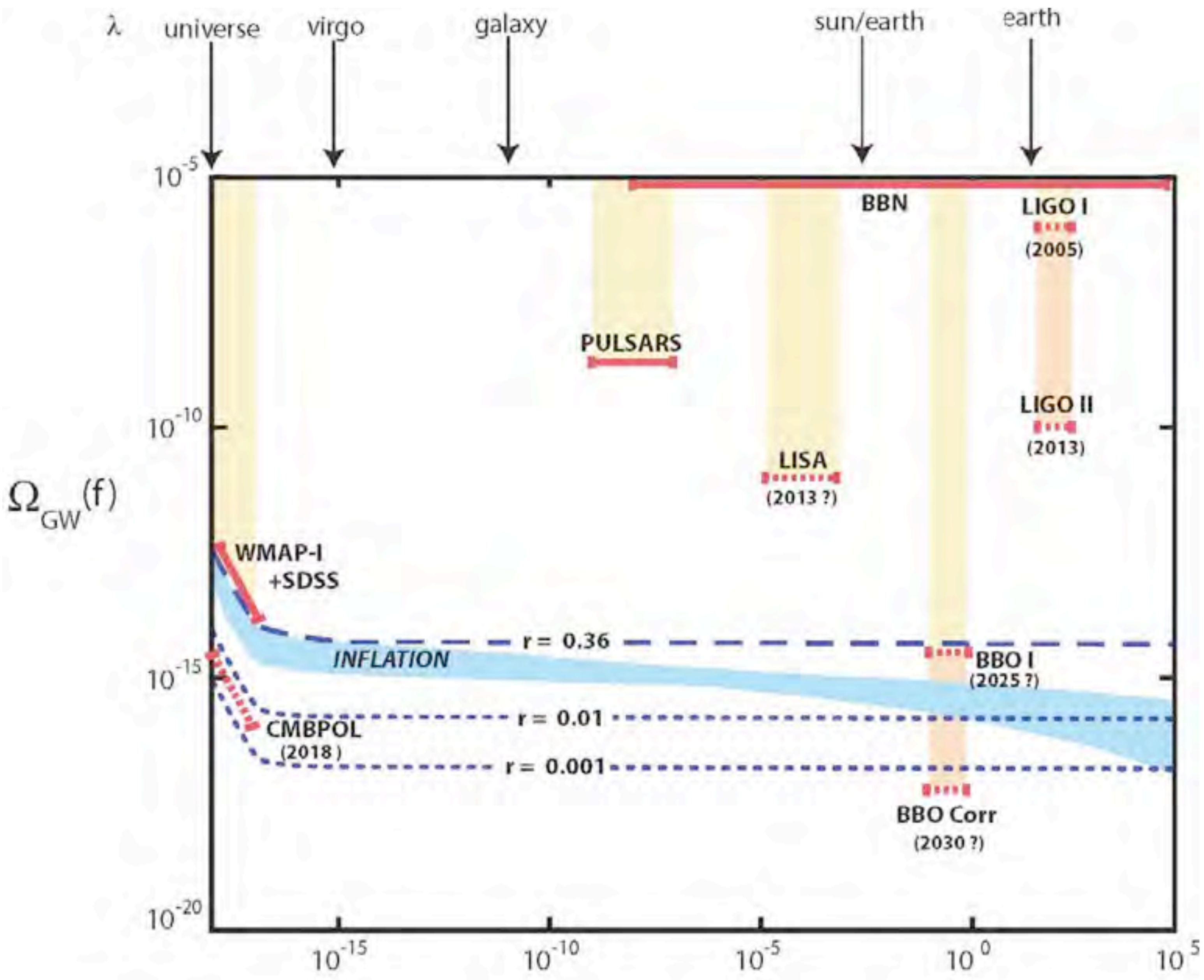
$$\boxed{I_{\mathrm{GW}}(f,\hat{\mathbf{\Omega}}) \;=\; \frac{4\pi^2 c}{3H_0^2}\,f^2\,H(f)\left[\mathcal{P}_+(\hat{\mathbf{\Omega}}) \;+\; \mathcal{P}_\times(\hat{\mathbf{\Omega}})\right]}$$

Theoretical Models



arXiv:astro-ph/0607043
D. Coward & T. Regimbau

SGWB Probes



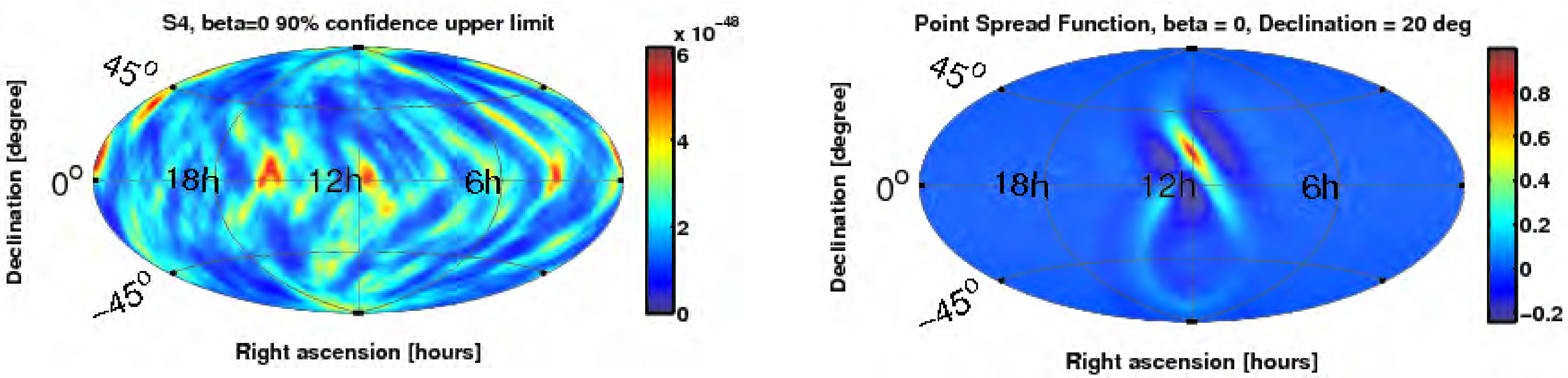
CMB Task Force Report

All-sky Upper Limits

- Constraint from LIGO: $\Omega_{\rm GW}(f) < 6.9 \times 10^{-6}$
  - best in the frequency range around 100Hz
- from WMAP 7 year (Larson et al): $\Omega_{\rm GW}\, h^2 \lesssim 6 \times 10^{-13}$
  - frequency 10^{-17} - 10^{-16} Hz
- structure formation (Smith et al): $\Omega_{\rm GW}(f)\, h^2 < 8.4 \times 10^{-6}$
  - frequency range 10^{-15} - 10^{-10} Hz
- Prediction from slow roll inflation: $\Omega_{\rm GW}(f) \sim 10^{-16} - 10^{-15}$

Directed Search Upper Limit

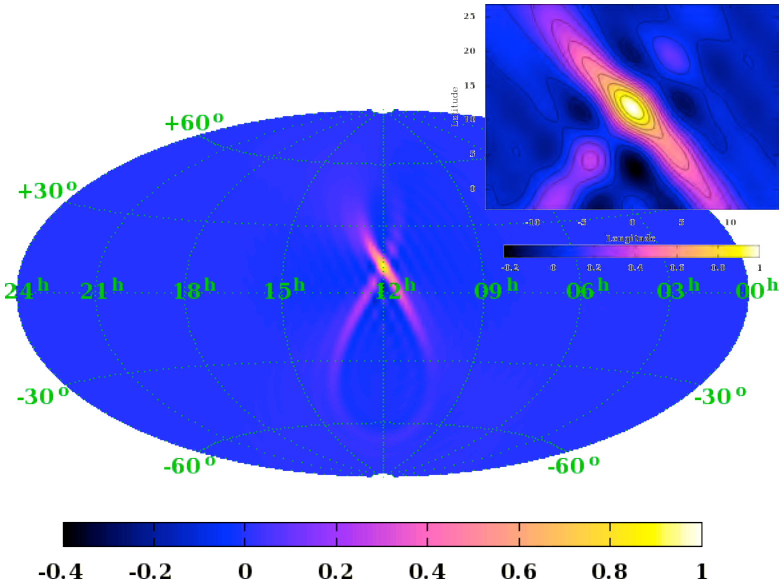
- Upper limit map from LIGO's 4th Science run



  - limits derived from dirty map
  - rigorous treatment requires deconvolution

Deconvolution of Directed GW Radiometer Map

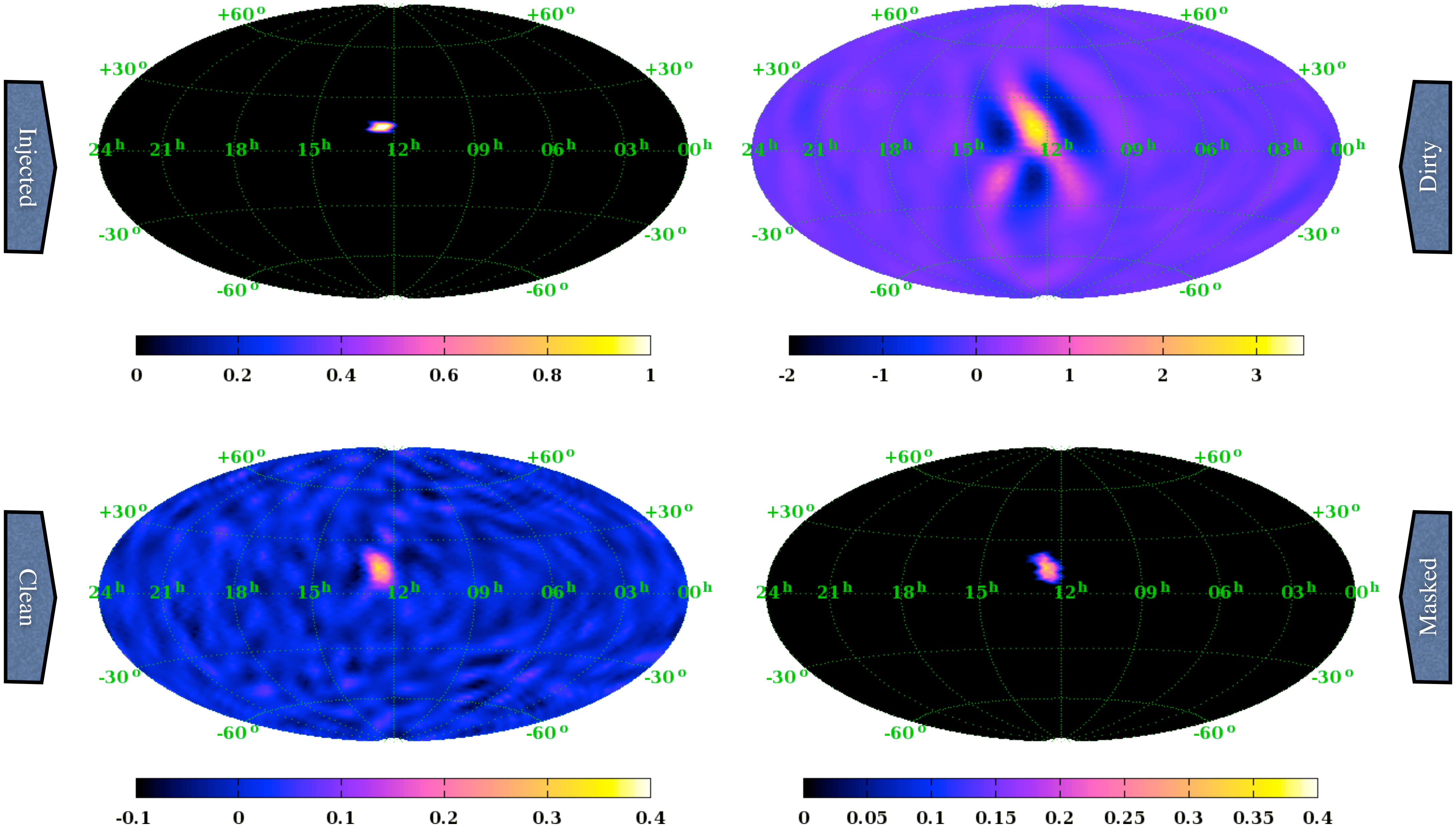
- Deconvolution is a challenge for any map making exercise
- The beam function was computed for each pixel
  - we used HEALpix pixelization (from CMB)



- Direct invert, solve convolution equation
  - we used Conjugate Gradient (CG) method (from CMB)
- Pixels below a certain threshold were masked
  - we used few times RMS of noise only clean map as threshold

Example of Directed Radiometer Deconvolution

- Toy 4-Pixel source near Virgo



Deconvolution in Spherical Harmonic Basis

- Noise Covariance matrix of the clean map
- Conjugate gradient method inverts an equation, not the kernel matrix
- Here we used SVD based regularization
  - ignore all the insensitive modes
  - or, set all of them to the cut-off value

$$\Gamma = USU^* \quad => \Gamma^{-1} = US^{-1}U^*$$

$$\forall S_i \in \mathbf{S}, \quad S_i^{-1} := \begin{cases} 1/S_i & \text{if } \mathrm{S_i} > \mathrm{S_{min}} \\ 0 \text{ or } 1/\mathrm{S_{min}} & \text{otherwise} \end{cases}$$